Constructing piece-wise-constant models using general measures in non-linear, minimum-structure inversion algorithms

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SUMMARY

General, non-sum-of-squares measures can be easily incorporated in standard minimum-structure, under-determined inversion algorithms via the iteratively re-weighted least squares technique. This means the benefits of minimum-structure, under-determined algorithms – robustness because of the dominance of minimizing model complexity, minimal influence of the starting model, and no influence of the model parameterization – can be retained and yet piece-wise-constant, blocky models constructed. In addition, the generalization enables measures of data misfit to be used that are more robust than a sum-of-squares measure when the noise in the observations is not Gaussian. The iteratively re-weighted least squares procedure involves a readily-implemented modification of the normal equations for the minimization of sum-of-squares measures, and the iterations required by this procedure can be directly amalgamated with those already needed because of the inherent non-linearity of the inverse problems we want to solve. The increased computational cost is not large, varying from 10% more iterations to twice as many iterations.

INTRODUCTION

The geophysical inverse problem is non-unique. Earth models constructed by inversion procedures must be consistent with any *a priori* information as well as reproduce the observations to an acceptable degree. For certain environments, the smeared-out, fuzzy features that are typical of models produced by minimum-structure algorithms do mimic the character of the subsurface. However, there are other regions where piece-wise-constant, blocky models would be a better representation. Such models can be constructed by minimizing measures of model structure other than the traditional sum-of-squares, or l_2, measure.

There have been a number of reports of the use of general measures of model structure in non-linear minimum-structure inversions: we have previously investigated their application to the one-dimensional inversion of time-domain electromagnetic data (Farquharson and Oldenburg, 1998); Portniaguine and Zhdanov (1999) applied a measure called a minimum support functional to three-dimensional inversions of gravity and magnetic data; and Oldenburg and Ellis (1993) used an l_1 norm (and linear programming) in a two-dimensional inversion of magnetotelluric data.

The use of general measures for some linear inverse problems has been reported: Oldenburg (1984) inverted lead isotope data using an l_1 norm of the model's gradient, and Sacchi and Ulrych (1996) obtained sparse solutions to Radon and Fourier transforms by treating them as inverse problems and minimizing a measure related to the solution's sparseness.

Certain non-sum-of-squares measures are more robust measures of misfit when the noise in a data-set is not Gaussian. Because of this, Gersztenkorn et al. (1986), for example, solve the one-dimensional seismic inverse problem using an l_1 measure of data misfit. Robust measures of misfit have also been used in the processing of magnetotelluric data (see, for example, Egbert and Booker, 1986)

Here we summarize the iteratively re-weighted least squares technique for a solution to non-linear inverse problems that uses general measures of model structure and data misfit. We illustrate the method with two-dimensional inversions of a synthetic direct-current resistivity data-set.

GENERAL MEASURES AND THE NON-LINEAR INVERSE PROBLEM

Consider the standard minimum-structure approach to solving an inverse problem which involves finding the model $\mathbf{m}$ that minimizes the objective function:

$$\Phi \;=\; \phi_d \;+\; \beta\,\phi_m, \tag{1}$$

where ϕ_d is a measure of data misfit:

$$\phi_d \;=\; \phi_d(\mathbf{u}), \tag{2a}$$

$$\mathbf{u} = \mathbf{W}_d(\mathbf{d}^{\rm obs} - \mathbf{d}^{\rm prd}), \tag{2b}$$

where $\mathbf{d}^{\rm obs}$ is the vector of observations, $\mathbf{d}^{\rm prd}$ is the vector of data computed for the model $\mathbf{m}$, and $\mathbf{W}_d$ is a diagonal matrix whose elements are the estimates of the standard deviations of the noise; where ϕ_m is a measure of the amount of structure in the model:

$$\phi_m \;=\; \alpha_s \phi_s(\mathbf{v}_s) \,+\, \alpha_x \phi_x(\mathbf{v}_x) \,+\, \alpha_z \phi_z(\mathbf{v}_z), \tag{3a}$$

$$\mathbf{v}_i = \mathbf{W}_i(\mathbf{m} - \mathbf{m}^{\mathrm{ref}}), \tag{3b}$$

where $i = s$, x and z for a two-dimensional model, $\mathbf{W}_s$ is a diagonal weighting matrix, $\mathbf{W}_x$ and $\mathbf{W}_z$ are the first-order finite-difference operators for the x- and z-directions, $\mathbf{m}^{\mathrm{ref}}$ is a reference model, and α_i are user-specified coefficients; and where β is the trade-off parameter. The general form of ϕ_d and ϕ_i is:

$$\phi(\mathbf{x}) \;=\; \sum_{j=1}^{N} \rho(x_j). \tag{4}$$

For a non-linear inverse problem, an iterative procedure is required in which the linear approximation of the relationship between the data and model parameters is used at each iteration. The situation at the nth iteration is therefore that of finding the model $\mathbf{m}^n$ that minimizes

$$\Phi^n \;=\; \phi_d^n \,+\, \beta^n \,\phi_m^n, \tag{5}$$

where

$$\phi_d^n \;=\; \phi_d(\mathbf{u}), \tag{6a}$$

$$\mathbf{u} = \mathbf{W}_d(\mathbf{d}^{\mathrm{obs}} - \mathbf{d}^{n-1} - \mathbf{J}\,\delta\mathbf{m}), \tag{6b}$$

where $\mathbf{d}^{n-1}$ is the vector of data for the model, $\mathbf{m}^{n-1}$, obtained from the previous iteration, $\delta\mathbf{m} = \mathbf{m}^n - \mathbf{m}^{n-1}$, and $\mathbf{J}$ is the Jacobian matrix of sensitivities for the linear approximation:

$$\mathbf{d}^n \;\approx\; \mathbf{d}^{n-1} \,+\, \mathbf{J}\,\delta\mathbf{m}, \tag{7a}$$

$$J_{ij} \;=\; \frac{\partial d_i}{\partial m_j}; \tag{7a}$$

and where

$$\phi_m^n \;=\; \alpha_s \phi_s(\mathbf{v}_s) \,+\, \alpha_x \phi_x(\mathbf{v}_x) \,+\, \alpha_z \phi_z(\mathbf{v}_z), \tag{8a}$$

$$\mathbf{v}_i = \mathbf{W}_i(\mathbf{m}^{n-1} + \delta\mathbf{m} - \mathbf{m}^{\mathrm{ref}}). \tag{8b}$$

The solution at the nth iteration involves differentiating Φ^n with respect to the elements of $\delta\mathbf{m}$ and equating the resulting expressions to zero. Differentiating the general form of the measures (eq. 4) gives

$$\frac{\partial \phi(\mathbf{x})}{\partial\,\delta m_k} \;=\; \sum_{j=1}^{N} \rho'(x_j) \frac{\partial\, x_j}{\partial\,\delta m_k}, \tag{9}$$

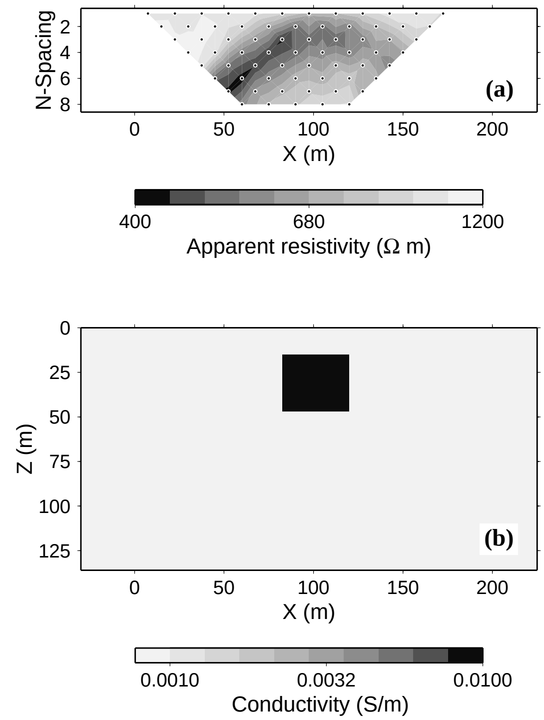

Figure 1. (a) The example synthetic DC resistivity data-set, and (b) the two-dimensional model from which it was computed.

that is,

$$\frac{\partial \phi(\mathbf{x})}{\partial\,\delta\mathbf{m}} \;=\; \mathbf{B}^T \mathbf{q}, \tag{10}$$

where $\partial\phi/\partial\,\delta\mathbf{m} \;=\; \big(\partial\phi/\partial\,\delta m_1, \,\ldots\,, \partial\phi/\partial\,\delta m_N\big)^T$, $B_{ij} \;=\; \partial x_i/\partial\,\delta m_j$, and $\mathbf{q} \;=\; \big(\rho'(x_1), \,\ldots\,, \rho'(x_N)\big)^T$. Eq. (10) can be reformulated by introducing a diagonal matrix:

$$\mathbf{R} \;=\; \mathrm{diag}\big\{\rho'(x_1)/x_1, \,\ldots\,, \rho'(x_N)/x_N\big\}, \tag{11}$$

which leads to

$$\frac{\partial \phi(\mathbf{x})}{\partial\,\delta\mathbf{m}} \;=\; \mathbf{B}^T \mathbf{R}\,\mathbf{x}. \tag{12}$$

For the measure of misfit, ϕ_d^n, in the objective function, $\mathbf{B}$ is $-\mathbf{W}_d\mathbf{J}$, and for the components, ϕ_i, of the measure of model structure, $\mathbf{B}$ is $\mathbf{W}_i$. The system of equations to be solved at each iteration is therefore

$$\Big[\,\mathbf{J}^T\mathbf{W}_d^T\mathbf{R}_d\mathbf{W}_d\mathbf{J} \,+\, \beta^n \sum_{i=1}^{3} \alpha_i \mathbf{W}_i^T\mathbf{R}_i\mathbf{W}_i\,\Big]\,\delta\mathbf{m} \\ = \mathbf{J}^T\mathbf{W}_d^T\mathbf{R}_d\mathbf{W}_d\big(\mathbf{d}^{\mathrm{obs}} - \mathbf{d}^{n-1}\big) \,+ \\ \beta^n \sum_{i=1}^{3} \mathbf{W}_i^T\mathbf{R}_i\mathbf{W}_i\big(\mathbf{m}_i^{\mathrm{ref}} - \mathbf{m}^{n-1}\big). \tag{13}$$

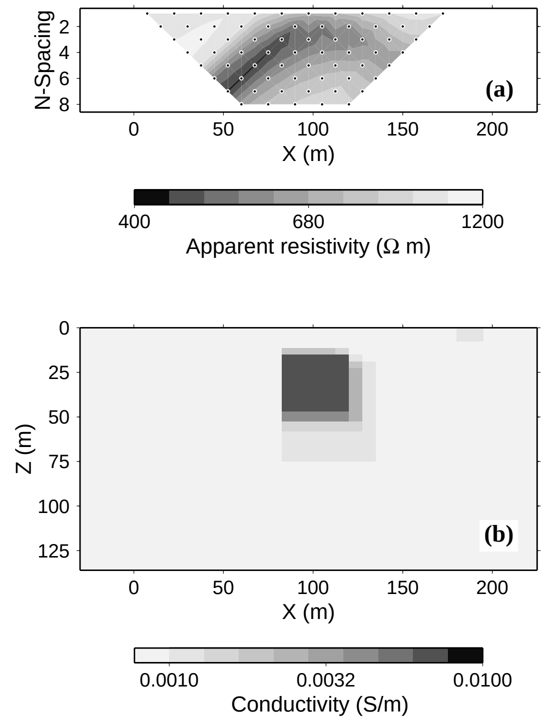

Figure 2. (a) The forward-modelled data, and (b) the constructed model for the inversion of the data-set in Figure 1(a) using Ekblom's measure with $p = 1$.

Except for the matrices $\mathbf{R}$, this system of equations is exactly the same as the normal equations that are obtained if sum-of-squares measures are used. The matrices $\mathbf{R}$ depend on the model, and so, like the Jacobian matrix, are updated at each iteration. This is the iteratively re-weighted least squares procedure.

There are numerous possibilities for the actual form of the measure used. For example, the l_p norm:

$$\|\mathbf{x}\|_p^p \;=\; \sum_{j=1}^{N} |x_j|^p, \tag{14}$$

(where $1 \le p < \infty$) results in piece-wise-constant models and is a robust measure of misfit when $p = 1$; the M-measure of Huber (1964):

$$\rho(x) \;=\; \begin{cases} x^2 & |x| \le c, \\ 2c|x| - c^2 & |x| > c, \end{cases} \tag{15}$$

(where c is the size of what is considered to be a small element of the vector) likewise gives piece-wise-constant models and is a robust measure of misfit; the perturbed p-norm-like measure of Ekblom (1987):

$$\rho(x) \;=\; \left(x^2 + \varepsilon^2\right)^{p/2}, \tag{16}$$

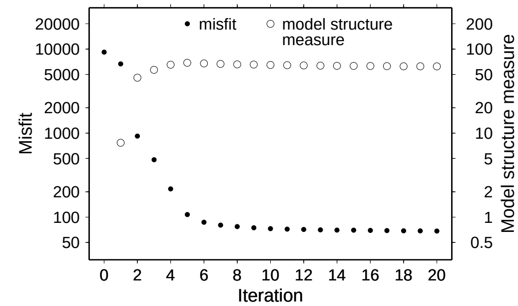

Figure 3. The variation of the misfit and measure of model structure during the inversion using Ekblom's measure with $p = 1$.

(originally with $1 \le p < \infty$, but extended to $0 \le p < 1$ by Zhang et al., 2000), is numerically nicer than the l_p norm (its derivative exists at $x = 0$); and the minimum support functional of Last and Kubik (1983) and Portniaguine and Zhdanov (1999):

$$\rho(x) \;=\; \frac{x^2}{x^2 + \varepsilon^2}, \tag{17}$$

which, for small ε, gives a measure proportional to the number of non-zero elements in the vector. The elements of the matrix $\mathbf{R}$ for the above measures are:

$$R_{ii} \;=\; \begin{cases} p\gamma^{p-2} & |x_i| \le \gamma, \\ p|x_i|^{p-2} & |x_i| > \gamma, \end{cases} \tag{18}$$

where γ is a small number so that $\mathbf{R}$ does not become singular as $x_i \to 0$;

$$R_{ii} \;=\; \begin{cases} 2, & |x_i| \le c, \\ 2c/|x_i|, & |x_i| > c; \end{cases} \tag{19}$$

$$R_{ii} \;=\; p\left(x_i^2 + \varepsilon^2\right)^{p/2-1}; \tag{20}$$

and

$$R_{ii} \;=\; \frac{2\varepsilon^2}{x^2 + \varepsilon^2}. \tag{21}$$

EXAMPLE

Figure 1 shows a two-dimensional conductivity model and a set of synthetic direct-current resistivity data (pole-dipole, with N-spacings of 1 to 8, giving 68 data in total) computed for the model. Gaussian noise of standard deviation equal to 5% of the value of a datum was added to give the data-set that was inverted.

Figure 2 shows the constructed model and the forward-modelled data resulting from the inversion of the data in Figure 1(a) using Ekblom's measure with

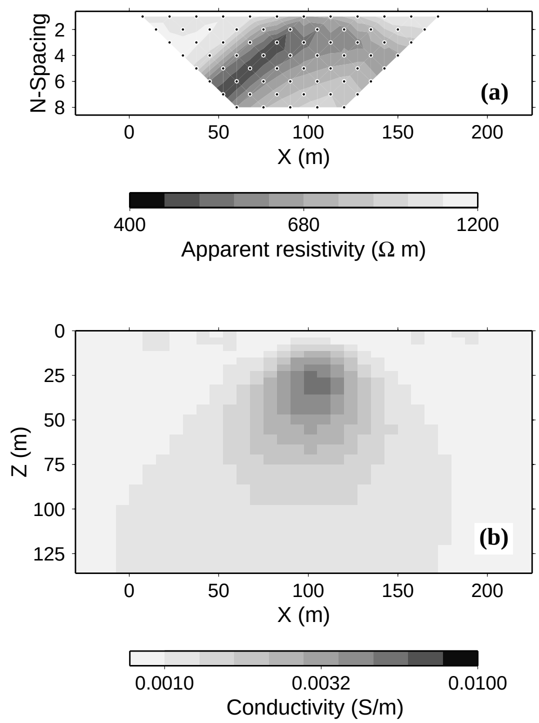

Figure 4. (a) The forward-modelled data, and (b) the constructed model for the inversion of the data-set in Figure 1(a) using Ekblom's measure with $p = 2$.

$p = 1$ (and $\varepsilon = 10^{-4}$) for the measure of model structure, and a sum-of-squares measure as the measure of the misfit (since, by construction, the noise in the data was Gaussian). A constant value of 10 was used for the trade-off parameter β. This gave a final misfit of 68. The changes in the misfit and in the measure of model structure during the course of the inversion are shown in Figure 3.

Figure 4 shows the results of using the sum-of-squares measure as the measure of model structure. Note the typical smeared-out nature of the conductive region. Figure 5 shows how the misfit and measure of model structure varied during this inversion. The trade-off parameter had a fixed value equal to 13. The final value of misfit was 66. Comparing Figures 3 and 5 shows that the inversion with Ekblom's measure and $p = 1$ required about 10 iterations to reach convergence whereas the inversion using the sum-of-squares measure required about 5 iterations.

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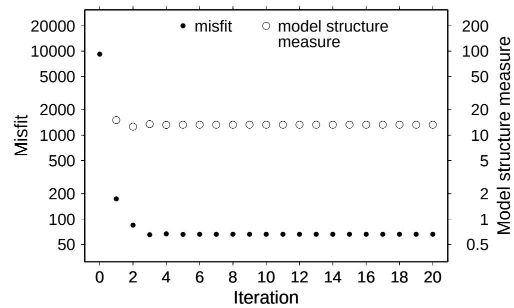

Figure 5. The variation of the misfit and measure of model structure during the inversion using Ekblom's measure with $p = 2$.

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