Gauged Vector Finite-Element Schemes for the Geophysical EM problem for Unique Potentials and Fields

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Outline

1. The Forward problem
   - E-field system
   - A-φ system

2. The uniqueness problem
   - The grounded wire and conductive prism example

3. Examples
   - Marine Hydrocarbon Modelling
     - Fields and Currents in the reservoir
   - MT example

4. Conclusions
The Forward problem

Calculating the electric field using the Helmholtz equation, E-field system

\[
\nabla \times \nabla \times E + i\omega \mu \sigma E - \omega^2 \mu \varepsilon E = -i\omega \mu J_e^s - \nabla \times J_m^s
\]

\[n \times E = 0\]

\(J_e^s\) and \(J_m^s\) are electric and magnetic source current densities.

Minimizing equation 1 over the physical domain \(\Omega\)
Discretization

Method of weighted residuals

\[ R = \int_{\Omega} \mathbf{W} \cdot \mathbf{r} \, d\Omega \]  \hspace{1cm} (2)

\( \mathbf{r} \) is the residual function.

Finite-element basis functions

\[ \tilde{\mathbf{E}} = \sum_{i=1}^{N_{\text{edges}}} \tilde{E}_i \mathbf{N}_i \]  \hspace{1cm} (3)

\( \mathbf{N}_i \) linear edge elements.

\[ \int_{\Omega} (\nabla \times \mathbf{W}) \cdot (\nabla \times \mathbf{N}_i) \, d\Omega + \int_{\Omega} (ik_1 - k_2)(\mathbf{W} \cdot \mathbf{N}_i) \, d\Omega = \int_{\Omega} \mathbf{W} \cdot \mathbf{S} \, d\Omega \]  \hspace{1cm} (4)

\[ \mathbf{S}(\mathbf{r}) + (ik_1 - k_2)\mathbf{M}(\sigma, \mathbf{r}) = \text{RHS} \]  \hspace{1cm} (5)
The Forward problem

E-field system

ill-conditioned system

\[
[S(r) + (i k_1 - k_2) M(\sigma, r)] \tilde{E} = \text{RHS}
\]

For small \(ik_1 - k_2\): nearly singular LHS

Non-smooth RHS

Iterative solver GMRES with ILU preconditioning

Residual norm vs. iteration number
Another example
H-dipole source, half space of 0.01 S/m

The residual norm is not small enough to give the correct field.
A-φ system: Remedy for the slow iterative problem

\[ \tilde{E} = -i\omega \tilde{A} - \nabla \tilde{\phi} \]  

(6)

The induction equation

\[ \nabla \times \nabla \times \tilde{A} + (i\omega \mu \sigma + \omega^2 \mu \epsilon) \tilde{A} + (\mu \sigma + i\omega \mu \epsilon) \nabla \tilde{\phi} = \mu \mathbf{J}_s \]  

(7)

Equation of conservation of charge

\[ -i\omega \nabla \cdot (\sigma \tilde{A}) - \nabla \cdot (\sigma \nabla \tilde{\phi}) + \omega^2 \nabla \cdot (\epsilon \tilde{A}) - i\omega \nabla \cdot (\epsilon \nabla \tilde{\phi}) = -\nabla \cdot \mathbf{J}_s \]  

(8)

Finite-element approximation of the potentials

\[ \tilde{A} = \sum_{i=1}^{N_{\text{edges}}} \tilde{A}_i \mathbf{N}_i \]  

\[ \tilde{\phi} = \sum_{k=1}^{N_{\text{nodes}}} \tilde{\phi}_k \mathbf{N}_k \]
System to solve

\[
\begin{pmatrix}
S + i\omega \mu M_1 + \omega^2 \mu M_2 & \mu F_1 + i\omega \mu F_2 \\
i\omega F_1^T + \omega^2 F_2^T & H_1 + i\omega H_2
\end{pmatrix}
\begin{pmatrix}
\tilde{A}
\tilde{\phi}
\end{pmatrix}
= \begin{pmatrix}
\mu_0 S_1 \\
S_2
\end{pmatrix},
\]

\( S = \int_\Omega \nabla \times N_i \cdot \nabla \times N_j \, d\Omega \)

\( M_2 = \int_\Omega \epsilon N_i \cdot N_j \, d\Omega \)

\( F_2 = \int_\Omega \epsilon N_i \cdot \nabla N_k \, d\Omega \)

\( H_1 = \int_\Omega \sigma \nabla N_k \cdot \nabla N_l \, d\Omega \)

\( M_1 = \int_\Omega \sigma N_i \cdot N_j \, d\Omega \)

\( F_1 = \int_\Omega \sigma N_i \cdot \nabla N_k \, d\Omega \)

\( D = -\int_\Omega \nabla N_k \cdot N_j \, d\Omega \)

\( H_2 = \int_\Omega \epsilon \nabla N_k \cdot \nabla N_l \, d\Omega \)
Fast convergence for $A-\phi$ system

E-dipole and half space
Fast convergence for $A-\phi$ system

H-dipole and half space
Uniqueness problem

The ungauged system

\[ \nabla \times \nabla \times \tilde{A} + (i\omega \mu \sigma + \omega^2 \mu \epsilon)\tilde{A} + (\mu \sigma + i\omega \mu \epsilon)\nabla \tilde{\phi} = \mu J^s \]  

(10)

\[ -i\omega \nabla \cdot (\sigma \tilde{A}) - \nabla \cdot (\sigma \nabla \tilde{\phi}) + \omega^2 \nabla \cdot (\epsilon \tilde{A}) - i\omega \nabla \cdot (\epsilon \nabla \tilde{\phi}) = -\nabla \cdot J^s \]  

(11)

\[ \tilde{E} = -i\omega \tilde{A} - \nabla \tilde{\phi} \]  

(12)

Grounded wire and conductive prism example, Frequency 3 Hz

Iterative solution, GMRES from SPARSKIT (Saad, 1990)

Direct solution, MUMPS (Amestoy et al., 2001)
The uniqueness problem

The grounded wire and conductive prism example

The ungauged system produces unique E and H

But non-unique $A$ and $\phi$

Calculated Potentials

Total fields

Ansari et al. (Memorial University)
Continuity study for the source of non-uniqueness

\[ \tilde{A} = \sum_{i=1}^{N_{edges}} N_i \]

\[ \nabla \cdot N = 0. \]

\[ \nabla \cdot A|_{\partial \Omega} \neq 0. \]

\[ A_1 \cdot n_1 = A_2 \cdot n_2 \]
Continuity study for the $A-\phi$ ungauged system

Noisy $A$

\begin{align*}
A_x (A/m) & \\
\n-2e^{-09} & 0 & 2e^{-09} \\
\text{Real} & \\

\n-2e^{-09} & 0 & 2e^{-09} \\
\text{Imaginary} & \\

\n-4e^{-06} & -2e^{-06} & 0 & 1e^{-07} & 2e^{-07} & 3e^{-07} \\
\x (m) & \\

\n800 & 900 & 1000 & 1100 & 1200 \\
\end{align*}
Leakage in $A$ causes non-unique $A$ and $\phi$
Total $E$ is unique

Iterative sln.

Direct sln.
Gauge fixing

Conventional method does not work!

\[
\nabla \times \nabla \times \vec{A} - \nabla(\nabla \cdot \vec{A}) + (i\omega \mu \sigma + \omega^2 \mu \epsilon)\vec{A} + (\mu \sigma + i\omega \mu \epsilon)\nabla \tilde{\phi} = \mu \mathbf{J}^s
\]

\[
\int_{\Omega} \mathbf{N}_i \cdot \nabla(\nabla \cdot \vec{A}) \, d\Omega = 0.
\]

Method 1: explicit gauging

\[
\nabla \times \nabla \times \vec{A} + (i\omega \mu \sigma + \omega^2 \mu \epsilon)\vec{A} + (\mu \sigma + i\omega \mu \epsilon)\nabla \tilde{\phi} = \mu \mathbf{J}^s
\]

\[
\begin{align*}
- i\omega \nabla \cdot (\sigma \vec{A}) + \nabla \cdot \vec{A} - \nabla \cdot (\sigma \nabla \tilde{\phi}) + \omega^2 \nabla \cdot (\epsilon \vec{A}) - i\omega \nabla \cdot (\epsilon \nabla \tilde{\phi}) &= -\nabla \cdot \mathbf{J}^s \\
\end{align*}
\]

\[
\begin{pmatrix}
\mathbf{S} + i\omega \mu \mathbf{M}_1 + \omega^2 \mu \mathbf{M}_2 \\
\end{pmatrix}
\begin{pmatrix}
\mu \mathbf{F}_1 + i\omega \mu \mathbf{F}_2 \\
\end{pmatrix}
\begin{pmatrix}
\vec{A} \\
\tilde{\phi}
\end{pmatrix}

= 
\begin{pmatrix}
\mu_0 \mathbf{S}_1 \\
\mathbf{S}_2
\end{pmatrix},
\]

\[
\mathbf{D} = \int_{\Omega} \mathbf{N}_k(\nabla \cdot \mathbf{N}_i) \, d\Omega
\]
Solutions - Gauged system

Grounded wire and prism example

113611 cells, 18669 nodes and 132507 edges.
Iterative solver, GMRES is slow even for large Krylov subspaces

Direct solver, MUMPS is used.
The uniqueness problem

The grounded wire and conductive prism example

Calculated Electric field

Real

Imaginary

|Ex| (A/m)

Ungauged, iterative sln.
Gauged, direct sln.

Ansari et al. (Memorial University) 3D Modelling October 20, 2015 19 / 51
Calculated Potentials

Real

Imaginary

<table>
<thead>
<tr>
<th>|A_x| (A/m)</th>
</tr>
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<tbody>
<tr>
<td>Real</td>
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<tr>
<td>Imaginary</td>
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</table>

<table>
<thead>
<tr>
<th>|V\phi_x| (A/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real</td>
</tr>
<tr>
<td>Imaginary</td>
</tr>
</tbody>
</table>

Ansari et al. (Memorial University)
Discontinuity fix, smooth vector potential

![Graphs showing real and imaginary parts of vector potential and electric field components.](image)
Another method for gauging

Method 2: Applying the Coulomb gauge condition directly and using a Lagrange multiplier in the induction equation

\[
\nabla \times \nabla \times \mathbf{A} + (i \omega \mu \sigma + \omega^2 \mu \epsilon) \mathbf{A} + (\mu \sigma + i \omega \mu \epsilon) \nabla \phi + \nabla \lambda = \mu \mathbf{J}^s \quad (14)
\]

\[
-i \omega \nabla \cdot (\sigma \mathbf{A}) - \nabla \cdot (\sigma \nabla \phi) + \omega^2 \epsilon \nabla \cdot \mathbf{A} - i \omega \epsilon \nabla \cdot \nabla \phi = -\nabla \cdot \mathbf{J}^s \quad (15)
\]

\[-\nabla \cdot \mathbf{A} = 0 \quad (16)\]

Finite-element approximation

Nodal elements for the Lagrange multiplier

\[
\tilde{\lambda} = \sum_{i=1}^{N_{\text{nodes}}} \tilde{\lambda}_i N_i \quad (17)
\]
The system to solve

\[
\begin{pmatrix}
S + i\omega \mu M_1 + \omega^2 \mu M_2 & \mu F_1 + i\omega \mu F_2 & \mu L \\
\mu F_1^T + \omega^2 F_2^T & H_1 + i\omega H_2 & 0 \\
L^T & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
\tilde{A} \\
\phi \\
\lambda \\
\end{pmatrix}
=
\begin{pmatrix}
\mu_0 S_1 \\
S_2 \\
0 \\
\end{pmatrix},
\]

(18)

\[
L = \int_{\Omega} \mathbf{N}_i \cdot \nabla \lambda_k \ d\Omega
\]
Lagrange multiplier system
A better iterative solution

Residual norm

Iteration number
Calculated Fields

Real

Imaginary

\[ |E_x| (\text{V/m}) \]

\[ x (\text{m}) \]

- Ungauged, iterative
- Aug. gauged, direct
- Lag. gauged, iterative
- Lag. gauged, direct
Calculated potentials

The uniqueness problem

The grounded wire and conductive prism example

Calculated potentials

Real

Imaginary

|A_x| (A/m)

|∇φ_x| (A/m)

Ansari et al. (Memorial University)
Discontinuity fix

\[ A_x (A/m) \]

\[ \nabla \Phi_x (A/m) \]

\[ E_x (A/m) \]

\[ x(m) \]
Unique fields

Augmented, direct

Lagrangian, iterative

Lagrangian, direct

\( i\omega A \) \( \nabla \phi \) \( E \)

\( \text{Re} \)
A realistic 3D model offshore Newfoundland, Canada - topographic reservoir is at $z \approx 1100$ m

Courtesy of Husky Energy
Another perspective

Air

Water

Reservoir

Sediments

Courtesy of Husky Energy

Ansari et al. (Memorial University)
A realistic 3D model- yz cross section

Courtesy of Husky Energy
### Physical properties

<table>
<thead>
<tr>
<th>Layer</th>
<th>Conductive (S/m)</th>
<th>Relative Permittivity (F/m)</th>
<th>Permeability (H/m)</th>
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</thead>
<tbody>
<tr>
<td>Air</td>
<td>$10^{-8}$</td>
<td>1</td>
<td>$\mu_0$</td>
</tr>
<tr>
<td>Sea water</td>
<td>3.3</td>
<td>80</td>
<td>$\mu_0$</td>
</tr>
<tr>
<td>Sediments</td>
<td>0.71</td>
<td>1</td>
<td>$\mu_0$</td>
</tr>
<tr>
<td>Reservoir</td>
<td>0.01</td>
<td>1</td>
<td>$\mu_0$</td>
</tr>
</tbody>
</table>

Courtesy of Husky Energy

### CSEM array

Electric dipole source of 200 m, Y-directed
Observation locations Y-axis
Frequency 0.25 Hz
Electric field response

|  $|E_y|$ (V/m) |
|---|
|  $y$ (m) |

- Ungauged
- Gauged, Lagrangian system
- Gauged, Augmented system
- Half space
Magnetic field response

- Ungauged
- Gauged, Lagrangian system
- Gauged, Augmented system
- Half space
Examples

Marine Hydrocarbon Modelling

Secondary response, normalized

- Ungauged
- Gauged, Lagrangian system
- Gauged, Augmented system

Ansari et al. (Memorial University)
## Computations:

<table>
<thead>
<tr>
<th>System</th>
<th>cells</th>
<th>unknowns</th>
<th>nnz</th>
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<tbody>
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<td>1,710,090</td>
<td>98,163,278</td>
</tr>
<tr>
<td>Aug. Gauged</td>
<td>643,477</td>
<td>1,710,090</td>
<td>98,163,278</td>
</tr>
<tr>
<td>Lag. Gauged</td>
<td>643,477</td>
<td>1,920,406</td>
<td>119,477,836</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>System</th>
<th>Krylov</th>
<th>Iterations</th>
<th>GMRES</th>
<th>time</th>
<th>MUMPS</th>
<th>time</th>
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<tr>
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<td>15 GB</td>
<td>5868 sec</td>
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<td>NA</td>
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<tr>
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<td>-</td>
<td>15 GB</td>
<td>-</td>
<td>36 GB</td>
<td>1112 sec</td>
</tr>
<tr>
<td>Lag. Gauged</td>
<td>800</td>
<td>20,000</td>
<td>17 GB</td>
<td>76237 sec</td>
<td>50 GB</td>
<td>1779 sec</td>
</tr>
</tbody>
</table>

Ansari et al. (Memorial University) October 20, 2015
Iterative solution is slow for the gauged systems

- ungauged solution
- gauged lagrangian solution
- gauged augmented solution
Inductive and Galvanic parts
Fields and currents at depth, xy cross section at $z = 1050$ m
Close view at $z = 1050$ m
Inductive current, Gauged solution removed parasitic behavior
Galvanic current

Ungauged

Re

Im

Gauged

Re

Im

$\log_{10} J \ (A/m^2)$

Ansari et al. (Memorial University)
Total current density

Ungauged

Gauged

Re

Im

\( \log_{10} J \, (\text{A/m}^2) \)
A Magnetotelluric example

COMMENI 3D-1A model
Meshed domain
Closer view

cells 526022, nodes 85752
Apparent resistivity for off-diagonal Z tensor

![Graphs showing apparent resistivity for off-diagonal Z tensor](image)

- Ungauged
- Gauged, lagrangian
- Farquharson and Miensopust, 2011
- Zhdanov et al., 1997

\[ \rho_a (\Omega \text{m}) \]

\[ x, y, z \]
Convergence

Method | nnz   | dimension | iterative soln. | Direct soln.
--- | --- | --- | --- | ---
Ungauged | 80491966 | 1395528 | 11 GB | NA
Gauged   | 122457534 | 1967204 | 13 GB | Not needed
Inductive part

Ungauged

Gauged

Re

Im

Re

Im

log_{10} E (V/m)

Gauged

Ungauged

Re

Im

log_{10} E (V/m)
Galvanic part

**Ungauged**

- 

**Gauged**

- 

\[ \text{Re} \]

\[ \text{Im} \]
Total field

Ungauged

Gauged

Ansari et al. (Memorial University)
Numerical forward modeling methods for explicitly gauging the vectorial finite element solution of the geophysical electromagnetic problem are presented.

The electric field is decomposed into vector and scalar potentials and subsequently used in the Helmholtz diffusion equation and the equation of conservation of charge.

The nonuniqueness difficulty is counteracted by explicitly incorporating the Coulomb gauge condition in the system: firstly by augmenting the equation of conservation of charge and secondly by considering the gauge condition in an individual equation.