Finite-volume modelling of geophysical electromagnetic data using potentials on unstructured staggered grids

Hormoz Jahandari and Colin G. Farquharson

Department of Earth Sciences
St. John’s, Newfoundland, Canada

SEG 85th Annual Meeting, New Orleans

October 20, 2015
Unstructured grids
Finite-volume discretization of Maxwell’s equations (direct EM-field and potential formulation)
Example for a grounded wire source
Example for a helicopter EM survey
Conclusions
Unstructured grids

- Model irregular structures
Unstructured grids

- Topographical features
- Geological interfaces
- Local refinement (at observation points, sources, interfaces)
Dual tetrahedral-Voronoi grids

- Grid generator: TetGen (Si, 2004)

**tetrahedral grid**

**Voronoi grid**
Staggered finite-volume schemes

- Magnetic field divergence free
- Easy for implementing boundary conditions
- Satisfies the continuity of tangential $E$
- Physically meaningful
Staggered finite-volume schemes

Dual tetrahedral-Voronoï grid

Delaunay-Voronoï contours

E-field

H-field
### Staggered finite-volume schemes

#### Direct EM-field method
- Unknowns are E and/or H
- Simpler
- Smaller system of equations
- Ill-conditioned

#### EM Potential \((A - \phi)\) method
- Unknowns are \(A\) and \(\phi\)
- Larger system of equations
- Well-conditioned
- Allows studying the galvanic and inductive parts
Direct EM-field formulation of Maxwell’s equations

- Maxwell’s equations:
  \[ \nabla \times \mathbf{E} = -i\omega\mu_0\mathbf{H} - i\omega\mu_0\mathbf{M}_p \]
  \[ \nabla \times \mathbf{H} = \sigma \mathbf{E} + \mathbf{J}_p \]

- Helmholtz equation for electric field
  \[ \nabla \times \nabla \times \mathbf{E} + i\omega\mu_0\sigma\mathbf{E} = -i\omega\mu_0\mathbf{J}_p - i\omega\mu_0(\nabla \times \mathbf{M}_p) \]

- Homogeneous Dirichlet boundary condition:
  \[ \mathbf{E} = 0 \quad \text{at} \ \infty \]
  or
  \[ \mathbf{E} \cdot \mathbf{\tau} = 0 \quad \text{on} \ \Gamma \]
Integral form of Maxwell's equations:

\[
\oint_{\partial S^D} E \cdot dl^D = -i\mu_0\omega \iint_{S^D} H \cdot dS^D - i\mu_0\omega \iint_{S^D} M_p \cdot dS^D
\]

\[
\oint_{\partial S^V} H \cdot dl^V = \sigma \iint_{S^V} E \cdot dS^V + \iint_{S^V} J_p \cdot dS^V
\]
Discretized form of Maxwell’s equations:

\[
\sum_{q=1}^{W_j^D} \sum_{l} E_{i(j,q)} I_{l(j,q)}^D = -i\mu_0 \omega H_j S_j^D - i\mu_0 \omega M_p S_j^D
\]

\[
\sum_{k=1}^{W_i^V} \sum_{p} H_{j(i,k)} I_{j(i,k)}^V = \sigma E_i S_i^V + J_{p} S_i^V
\]
Discretized form of Helmholtz equation:

\[
\sum_{k=1}^{W_i^V} \left( \left( \sum_{q=1}^{W_j^D} E_{i(q,j)} I_{i(j,q)} \right) \frac{I_{j(i,k)}^{V}}{S_{j(i,k)}^{D}} \right) + i\omega \mu_0 \sigma E_i S_{i}^{V}
\]

\[
= -i\omega \mu_0 \sum_{k=1}^{W_i^V} M_{p(j,i,k)} \frac{I_{j(i,k)}^{V}}{S_{j(i,k)}^{D}} - i\omega \mu_0 J_{p(i)}
\]
EM potential \((A-\phi)\) formulation of Maxwell’s equations

- Magnetic vector and electric scalar potentials:
  \[
  \mathbf{E} = -i\omega \mathbf{A} - \nabla \phi \\
  \mu_0 \mathbf{H} = \nabla \times \mathbf{A}
  \]

Helmholtz equation in terms of potentials with Coulomb gauge

\[
\nabla \times \nabla \times \mathbf{A} - \nabla (\nabla \cdot \mathbf{A}) + i\omega \mu_0 \sigma \mathbf{A} + \sigma \mu_0 \nabla \phi = \mu_0 \mathbf{J}_p + \mu_0 \nabla \times \mathbf{M}_p
\]

Conservation of charge

\[
-\nabla \cdot \mathbf{J} = \nabla \cdot \mathbf{J}_p \\
i\omega \nabla \cdot \sigma \mathbf{A} + \nabla \cdot \sigma \nabla \phi = \nabla \cdot \mathbf{J}_p
\]

- Homogeneous Dirichlet boundary condition:
  \[
  (\mathbf{A}, \phi) = 0 \quad \text{at } \infty
  \]
  or
  \[
  (\mathbf{A} \cdot \mathbf{\tau}, \phi) = 0 \quad \text{on } \Gamma
  \]
EM potential \((A-\phi)\) formulation of Maxwell’s equations

Relations used for the finite-volume discretization

\[
\nabla \times \mathbf{H} - \mu_0^{-1} \nabla \psi + i \omega \sigma \mathbf{A} + \sigma \nabla \phi = \mathbf{J}_p + \nabla \times \mathbf{M}_p \tag{1}
\]

\[
\mu_0 \mathbf{H} = \nabla \times \mathbf{A} \tag{2}
\]

\[
\psi = \nabla \cdot \mathbf{A} \tag{3}
\]

\[
i \omega \nabla \cdot \sigma \mathbf{A} + \nabla \cdot \sigma \nabla \phi = \nabla \cdot \mathbf{J}_p \tag{4}
\]
The system of equations for the A-$\phi$ method

- Decompose $A$ and $\phi$ into real and imaginary parts:
  \[ A = A_{re} + iA_{im} \quad ; \quad \phi = \phi_{re} + i\phi_{im} \]

- Resulting block matrix equation:
  \[
  \begin{pmatrix}
  A & -\omega B & C & 0 \\
  \omega B & A & 0 & -C \\
  0 & -\omega D & E & 0 \\
  \omega D & 0 & 0 & E \\
  \end{pmatrix}
  \begin{pmatrix}
  A_{re} \\
  A_{im} \\
  \phi_{re} \\
  \phi_{im} \\
  \end{pmatrix}
  =
  \begin{pmatrix}
  S_1 \\
  0 \\
  S_2 \\
  0 \\
  \end{pmatrix}
  \]

Sparse coefficient matrix, A-\phi method
Solution of the finite-volume schemes

- **Direct solution:** MUMPS sparse direct solver (Amestoy et. al, 2006)

- **Iterative solution:** BiCGSTAB and GMRES solvers from SPARSKIT (Saad, 1990)
Example 1: grounded wire

- 100 m wire along the x axis operating at 3 Hz
- Dimensions of the prism: 120 x 200 x 400 m
- $\sigma_{\text{ground}} = 0.02 \, \text{S/m}$; $\sigma_{\text{prism}} = 0.2 \, \text{S/m}$
- Observation points along the x axis
Example 1: grounded wire

- Dimensions of the domain: $40 \times 40 \times 40 \text{ km}$
- Number of tetrahedra: 162,689
Example 1: grounded wire

(on Apple Mac Pro; 2.26 GHz Quad-Core Intel Xeon processor)
Example 1: grounded wire

- Scattered electric field

In phase: $E_x (V/m)$

Quadrature: $E_x (V/m)$
Example 1: grounded wire

- **Galvanic part** \((-\nabla \phi)\)

- **Inductive part** \((-i\omega A)\)

- **Total electric field:**
  \[ E = -\nabla \phi - i\omega A \]
Example 1: grounded wire

- **Galvanic part** \((-\sigma \nabla \phi)\)

- **Inductive part** \((-i\omega \sigma A)\)

- **Total current density:**
  \[ J = -\sigma \nabla \phi - i\omega \sigma A \]
Example 1: grounded wire

- Cumulative error versus the changing cell size at the observation points

![Graph showing cumulative error versus cell size for different methods and orders.]
Example 1: grounded wire

<table>
<thead>
<tr>
<th></th>
<th>BCGSTAB</th>
<th>GMRES</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>with Coulomb gauge</td>
<td>without Coulomb gauge</td>
</tr>
<tr>
<td>b</td>
<td>without Coulomb gauge</td>
<td>with Coulomb gauge</td>
</tr>
<tr>
<td>c</td>
<td>with Coulomb gauge</td>
<td></td>
</tr>
<tr>
<td>d</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example 2: Ovoid, HEM survey

- Ovoid: massive sulfide ore body, Voisey’s Bay, Labrador, Canada
- HEM survey of the region has been simulated
- Transmitter and receiver towed below the helicopter 30 m above ground
- Transmitter-receiver separation was 8 m and the frequencies were 900 and 7200 Hz
- $\sigma_{\text{ground}} = 0.001$ and $\sigma_{\text{ovoid}} = 100$ S/m were chosen by try-and-error
- Number of tetrahedra: 240, 692
Example 2: Ovoid, HEM survey

Topography of the region
Example 2: Ovoid, HEM survey

- White dots show the observation points
Example 2: Ovoid, HEM survey

- The plan view
Example 2: Ovoid, HEM survey

- Grid refined at the sources and observation points
Example 2: Ovoid, HEM survey

- FV results (circles) vs real HEM data (lines)

![Graph showing HEM survey results](image)

- Dashed line: HEM; in phase
- Full line: HEM; quadrature
- Blue: 900 Hz
- Red: 7200 Hz

Northing (m)

$H_z$ (ppm)
Example 2: Ovoid, HEM survey

- Amplitude of the horizontal component of total current density at a vertical section along the observation profile
Galvanic part \((-\sigma \nabla \phi)\)

Inductive part \((-i \omega \sigma A)\)

Total current density:
\[ J = -\sigma \nabla \phi - i \omega \sigma A \]
Conclusions

- A finite-volume approach was used for modelling total field EM data. It used staggered tetrahedral-Voronoï grids.
- The potential formulation of Maxwell’s equation was discretized and solved and compared to the solution of the EM-field formulation.
- Accuracy and versatility were tested using two examples: one with a grounded wire source and a small conductivity contrast; another one with a realistic body, magnetic sources and a large conductivity contrast.
- Unlike the EM-field scheme, the $A - \phi$ scheme could be solved using generic iterative solvers.
- The gauged problems were harder to solve than the ungauged problems.
- Solutions were decomposed into galvanic and inductive parts. The results were in good agreement with the type of sources that were used.
- While both EM-field and potential schemes possessed the same trends of accuracy, the potential approach showed lower cumulative errors.
Acknowledgements

- ACOA  
  (Atlantic Canada Opportunities Agency)

- NSERC  
  (Natural Sciences and Engineering Research Council of Canada)

- Vale


