PHYSICAL REVIEW E **70**, 015601(R) (2004)

Extracting the Green function from diffuse, equipartitioned waves

Alison E. Malcolm* and John A. Scales
Physical Acoustics Laboratory and Center for Wave Phenomena, Department of Geophysics, Colorado School of Mines, Golden, Colorado 80401, USA

Bart A. van Tiggelen
Laboratoire de Physique et Modélisation des Milieux Condensés, CNRS, Maison des Magistères, Université Joseph Fourier, BP 166, 38042 Grenoble Cedex 9, France

(Received 25 November 2003; published 23 July 2004)

A ballistic pulse launched in a strongly scattering random medium becomes diffusive after a few mean-free times. In this regime of diffusive propagation there is a net flux of energy away from the source. Eventually the flux goes to zero, in the *equipartitioned* regime, in which the signal consists of equal amounts of energy propagating in all directions. In this regime the two-point, two-time correlation of the wave-field should equal the sum of the advanced and retarded Green functions associated with the average medium. We observe the emergence of the Green function from this correlation at about 9 mean-free times in a highly heterogeneous rock sample.

DOI: 10.1103/PhysRevE.70.015601 PACS number(s): 43.35.+d, 42.68.Ay, 62.80.+f, 91.60.Lj

Conventional imaging in media with strong scattering, such as rocks, can be difficult because coherent signals (e.g., directly arriving waves) are strongly attenuated by the scattering [1]. Theoretically it is possible to obtain imaging information from averages (over the randomness of the medium) of the correlation of the diffuse field [2]. In practice, however, it can be difficult to separate the diffuse field from the ambient noise of the experiment. The complicated nature of rock structure makes this particularly difficult since—although we believe the scattering structure of the rock to be random—we have no knowledge of the distribution or nature of the scatterers.

In 1950 Wiesner and Lee [3] showed that the Green function (GF) of any linear system can be obtained by driving the system with random noise and cross-correlating the noise input with the output. In helioseismology, Rickett and Claerbout [4] and Duval *et al.* [5] have used a related idea to synthesize acoustic waves inside the sun. More recently, Lobkis and Weaver [2] showed how the GF can be retrieved from the correlations of an equipartitioned (EP) thermal phonon field using a modal expansion, and used this idea [2,6] to compute the elastic GF. Campillo and Paul [7] used the same idea to extract the GF from the seismic coda. Here we pursue a laboratory experiment using a coarse-grained rock as a random medium in which the strong multiple scattering generates the necessary EP and study the time symmetry of the correlation function.

Imagine an ultrasonic pulse launched from the origin in the x-direction of an infinite random medium. Assume that intrinsic attenuation is weak relative to scattering so that the whole range of multiple-scattering behavior can be seen before the signal is dominated by noise. Define a *scan region S* some distance from the origin in which one measures the elastic wave-field $u(\mathbf{r},t)$. Before the source energy has reached the scan region, the elastic intensity in S, $\int_S |u|^2 d^3\mathbf{r}$, is equal to some background or ambient level. Once the source pulse enters S, the intensity shoots up and then begins to decay once the pulse has left S. We refer to this spike in intensity (associated with non-scattered energy) as the ballistic regime. After several mean-free times, the intensity begins to decay diffusively with time, as multiple scattering slows the transport of energy out of the scan region. This is the *diffusion regime*. Later, after many mean-free times, the flux of energy out of S falls to zero, but the energy averaged over S is still above the background level. This is the *equipartitioning regime* since the wave-field has no preferred wavenumber. This is in contrast with the equiparitioning among the P and S polarizations that sets in much earlier [8–10]. Eventually, intrinsic attenuation dominates and the energy falls to the noise level.

In the Physical Acoustics Laboratory at CSM we study ultrasonic wave propagation in heterogeneous media such as rocks and engineered composites using non-contacting ultrasonic laser sources and detectors. See Ref. [11] for an overview of our laboratory setup. The results shown here come from an experiment performed on a 135×54 mm cylinder of coarse-grained Llano granite. Although structures such as layering and large mineral inclusions are common in rocks, this particular rock is a relatively uniform mixture of grains averaging about 3–5 mm in diameter. In Ref. [11] we showed that the mean-free time in this type of rock around 1 MHz, averaged over bulk and surface waves, is about 2.9 μs (mean-free path about 9 mm) and that the average intensity is well fit by a diffusion model after a few mean-free times. We collected data at 5 locations ranging from 5 to 25 mm from the source position; the source-detector configuration is shown in Fig. 1(a). Fig. 1(b) shows an example of the data collected. To generate the necessary ensemble of measurements, we placed the sample on a rotational stage and collected data every 5°, on nine lines separated vertically by about 3 mm.

*URL: http://acoustics.mines.edu
Electronic address: amalcolm@dix.mines.edu

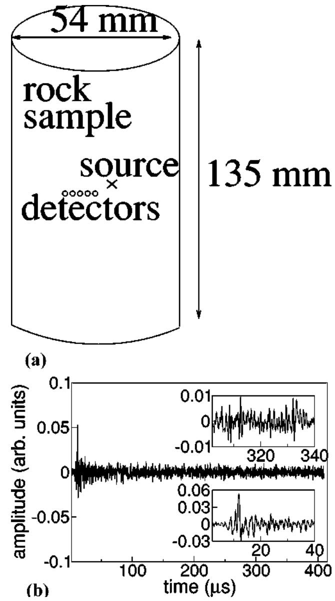

FIG. 1. (a) Experimental setup, 5 detectors starting 5 mm from the source with 5 mm spacing. (b) Typical data from the experiment. The lower inset shows the first arrivals and the upper inset shows a portion of the multiply scattered signal used in this experiment.

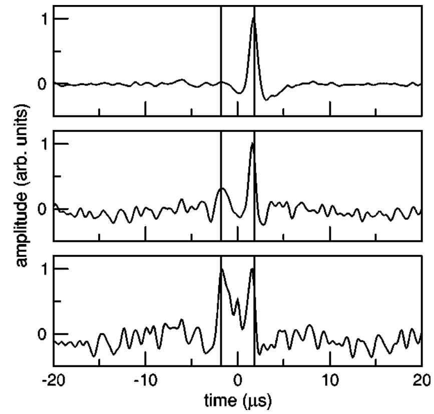

FIG. 2. Time correlation function in three different 30 μs windows of the recorded data. Data from the two detectors furthest from the source are used. The time used in the top panel (beginning at 5 μs, $R/ct=0.40$) is too early to see the acausal component, and the correlation is asymmetric. Between the times in the middle panel (beginning at 12.5 μs, $R/ct=0.29$) and the last panel (beginning at 25 μs, $R/ct=0.20$), the field becomes equipartitioned. The vertical lines indicate the expected arrival time of the surface wave, based on independent surface-wave velocity estimates.

When energy is traveling diffusively in a medium, a net energy flux exists in the direction of propagation. Once the field is EP there is no net energy transport and so the energy flux is zero. Thus, we shall characterize the EP regime as the time between when the flux decays to zero and that when the signal has relaxed to the ambient noise level. The intensity is dominated by surface waves because body waves are more rapidly attenuated than surface waves and because we record the vertical component of particle velocity at the surface [12]. From analysis of the intensity measurements we find that the signal has relaxed to the ambient level after about 100 μs and the spatial gradient of this intensity has decayed to zero by 15 μs, indicating that the EP regime is located between 15 and 100 μs.

The ensemble-averaged time-correlation function measured at two detector positions 5 mm apart at the surface was investigated in the ballistic, the diffusive and in the EP regime (Fig. 2). The transition from ballistic to diffusive propagation is clearly visible [13]. In all three regimes the Rayleigh wave could be identified from its arrival time but the time symmetry is different, contrary to measurements made outside the medium [14]. At early times, the majority of the energy is propagating away from the source. After about 25 μs, once the field is EP, equal amounts of energy propagate in all directions [10]. Thus, the observed correlation of the field will be symmetric about zero time when the field is EP and asymmetric when the propagation is ballistic or diffusive. The theory discussed below shows R/ct to be a dimensionless measure of how "late" the time window is. The quantity R is the source-detector distance (here the distance to the point halfway between the two detector positions is used). We approximate the velocity c by the Rayleigh wave velocity (2.8 mm/μs), which should be close to the energy velocity. Finally, t is the center time of the window used in the computations.

To explain the emergence of symmetry we briefly discuss the underlying theory of the EP of waves in disordered media. We consider the simplest case: Waves with no dispersion, no absorption, no polarization and propagating diffusively in an infinite medium, and *one* source with flat power spectrum over some bandwidth. We write $u(\mathbf{r},t)$ for the "displacement," which obeys the scalar wave equation

$$\frac{\omega^2}{c^2(\mathbf{r})}u(\mathbf{r},\omega) + \Delta u(\mathbf{r},\omega) = \mathcal{S}(\mathbf{r},\omega). \tag{1}$$

Transport theory makes the following statement for its autocorrelation function at two different frequencies, $\omega \pm \frac{1}{2}\Omega$, and at two different wave numbers, $\mathbf{k}\pm\frac{1}{2}\mathbf{q}$, [15],

$$\left\langle u\left(\omega+\frac{1}{2}\Omega,\mathbf{k}+\frac{1}{2}\mathbf{q}\right)u\left(\omega-\frac{1}{2}\Omega,\mathbf{k}-\frac{1}{2}\mathbf{q}\right)^*\right\rangle = \frac{\mathcal{S}(\omega)}{-i\Omega+Dq^2}\frac{\Delta G(\omega,\mathbf{k})}{i\omega}\left[1+\frac{c}{\omega}i\ell\mathbf{k}\cdot\mathbf{q}+\ldots\right]. \tag{2}$$

Here, $\Delta G(\omega,\mathbf{k}) = G(\omega+i\epsilon,\mathbf{k}) - G(\omega-i\epsilon,\mathbf{k})$ is the difference between the retarded and the advanced GF of the displacement of the host medium, ℓ is the mean free path, and $\mathcal{S}(\omega)$ is the power spectrum of the source, assumed point-like. The formula above expresses three important features. The first factor states that macroscopic transport, described by the parameters Ω and $\mathbf{q}$, obeys a diffusion equation with diffusion constant $D=\frac{1}{3}c\ell$ and source $\mathcal{S}(\omega)\delta(\mathbf{r})$. The micro-physics is contained in the second factor $\Delta G: -i\omega\Delta G(\omega,\mathbf{k})$, known as the spectral function of the medium [15], and counts the number of microscopic states available at frequency ω and wavenumber $\mathbf{k}$. For $\mathbf{q}=\mathbf{0}$ we conclude that the energy spectrum of the diffuse waves is the product of the power spectrum of the source and the spectral function. As a result, at given frequency, the different possible modes are EP. Finally, the $(\mathbf{k}\cdot\mathbf{q})$ term indicates that a macroscopic diffuse gradient, expressed by $\mathbf{q}$, can locally destroy this microscopic EP.

We can now easily translate back to space-time. Let us denote by $C(\mathbf{x},\tau)$ the *velocity* autocorrelation function $\langle \partial_t u(\mathbf{r}+\frac{1}{2}\mathbf{x},t+\frac{1}{2}\tau)\partial_t u(\mathbf{r}-\frac{1}{2}\mathbf{x},t-\frac{1}{2}\tau)\rangle$ as measured in this paper. We find from (2),

$$C(\mathbf{x},\tau) = \int \frac{d\omega}{2\pi}\int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}-i\omega\tau}[-i\omega + ic\ell(\mathbf{k}\cdot\partial_{\mathbf{r}})]\rho_\omega(\mathbf{r},t)\Delta G(\omega,\mathbf{k}), \quad (3)$$

with $\rho_\omega(\mathbf{r},t)$ the diffuse displacement energy at time t and at distance $\mathbf{r}$ from the source. If we assume that both the source power function and the scattering properties hardly vary over the chosen bandwidth, then the above expression simplifies to

$$C(\mathbf{x},\tau) = [\partial_\tau + c\ell\partial_{\mathbf{x}}\cdot\partial_{\mathbf{r}}]\rho(\mathbf{r},t)[G(\mathbf{x},\tau) - G(\mathbf{x},-\tau)] \quad (4)$$

This formula relates the velocity auto-correlation to the (band-limited) GF G of the displacement. The time-derivative generates a *time-symmetric* contribution —recognized as the *velocity* GF —expressing the perfect EP field; the spatial derivative gives an antisymmetric contribution, proportional to the spatial gradient of the intensity. By insertion of the typical solution $\rho\sim\exp(-R^2/4Dt)$ of the diffusion equation we can check that the ratio α of the causal $(\tau>0)$ to the anti-causal term $(\tau<0)$ is of order $\alpha=(1+3R/2ct)/(1-3R/2ct)$ at the elapsed time t. This analysis shows that the correlation function is very asymmetric at early times but that the asymmetry should slowly fade away with time.

Equation (4) is valid for a three-dimensional (3D) infinite random medium and is overly simplified to apply to a heterogeneous rock with surface detection. In optics the transition from diffuse to ballistic motion at the boundaries is a very complex problem [16]. Here we have the additional complication that two-dimensional (2D) surface waves exist that can mode convert to 3D bulk waves that all have different speeds and mean free times. The EP principle states that shear bulk waves largely dominate phase space [8–10], so that the diffusion constant $D\approx\frac{1}{3}V_S\ell_S$, where V_s is the shear wave speed and ℓ_S is the shear wave mean free path. In a surface measurement, however, surface waves still dominate [12]. By the absence of a 3D elastic diffusion theory with free-surface detection we have considered a quasi-2D mode-diffusion model to get a more quantitative picture [17]. We find that the asymmetry of the correlation function for the Rayleigh wave is given by $\alpha(\delta)=(1+1.18\delta R/ct)/(1-1.18\delta R/ct)$ that now contains a new factor $\delta\equiv D_R/D_S$ describing the ratio of the diffusion constant of the Rayleigh waves and the shear diffusion constant. We note that this study of time asymmetry could provide an estimate for the diffusion constant D_R that is usually not measured.

In Fig. 2(b), the ratio of the causal to anti-causal peak is $\alpha_{\mathrm{exp}}=1/0.32=3.13$. In this figure, R/ct declines from 0.6 to 0.2, giving theoretical ratios $\alpha_{\mathrm{theo}}(\delta=1)$ ranging from 6.4 to 1.6. This illustrates the sensitivity of α to the elapsed time which varies considerably inside the windows used for the correlation. As a result a quantitative comparision between theory and experiment to retrieve δ is inconclusive. If we adopt the value $R/ct=0.2$ at late times, this would yield a value of $\delta=2.2$ to get agreement with theory. This would imply a diffusion coefficient (and thus a mean free path) roughly twice as large for surface waves as for shear waves.

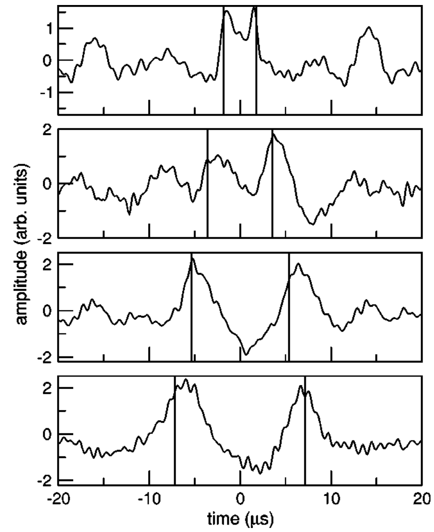

FIG. 3. Time correlation function for different offsets, each using 3 windows 30 μs in length. Top panel: Offset 5 mm, windows starting at 17.5 μs, $R/ct=0.1$. Second panel: Offset 10 mm, windows starting at 30, 120, and 190 μs, $R/ct=0.06$. Third panel: Offset 15 mm, windows starting at 80 μs with $R/ct=0.13$. Bottom panel: Offset 20 mm, windows starting at 157.5 μs, with $R/ct=0.03$. To ensure that each window represents an independent member of the ensemble, a 5 μs gap is left between consecutive windows.

In the recovered GF the surface wave arrives at times much smaller than the time-span of the portion of the data exhibiting EP. We can exploit this fact to increase the number of ensemble measurements used to construct the GF [19], by using multiple windows. This results in the GFs shown in Fig. 3. This figure confirms the expected linear change in arrival time with detector-detector offset. The fact that the surface waves can still be identified at distances as large as 20 mm implies that their mean-free-path ℓ_R is larger than the 9 mm estimated for the bulk ℓ_S from a fit to radiative transfer [11]. This is again consistent with $\delta\approx2$ implying $\ell_R\approx18$ mm.

From intensity measurements we find that the signal has decayed to the ambient noise level after about 100 μs. Some of the GFs in Fig. 3 use much later times, however. This indicates that either we are too conservative in our noise level estimate or that the noise itself is EP, allowing us to extract the GF even directly from the noise. Even with EP noise, we do not expect to extract the GF from the noise in our experiment, since the different measurements are taken at different times. We have independently measured the quality factor of the sample to be about 80, indicating that at late times we should be in the noise. Extracting ballistic waves from EP noise is a fascinating result that requires further study.

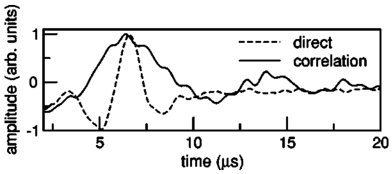

FIG. 4. Comparison of the retrieved Green function and the directly measured Green function; both are normalized to their peak amplitudes. They agree kinematically although high frequencies are suppressed in the retrieved Green function.

In Fig. 4 we compare the retrieved detector-detector GF to a direct measurement of the GF using an active source. What is most notable is the lower frequency content of the former. A very likely explanation for this is that the spectrum of the source decays with frequency, as we know is the case in our experiment. This affects the retrieved GF more than the directly measured one.

Because correlation is formally identical to time-reversal [18] we expect that only the phase plays a role in extracting the GF through correlations of the field. We confirm this by comparing the results above [Fig. 3(c)] with correlations obtained from the same data reduced to the sign bit shown in Fig. 5.

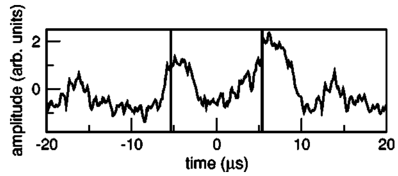

FIG. 5. Time correlation using only the sign bit of the data.

In conclusion, we have retrieved the ensemble averaged GF for ultrasonic waves traveling between two *detectors* in a strongly scattering medium by exploiting the multiple-scattering. Through the observed symmetry of the retrieved GF, we are able to determine whether or not the field itself is EP. We have also shown that the results hold when single bit data are used in place of higher precision measurements. A future direction of this work is to extract the GF in an even more complex medium, such as one containing an object to be imaged.

ACKNOWLEDGMENTS

We acknowledge many useful discussion with R. Snieder, K. van Wijk, A. Grêt, M. Haney, H. Douma, and C. Pacheco of CSM and M. Campillo, E. Larose and L. Margerin of LGIT. This work was partially supported by the National Science Foundation (EAR-0111804 and EAR-0337379), the U.S. Army Research Office (DAAG55-98-1-0070) and ACI 2066 of the French Department for Research.

[1] A. Tourin, M. Fink, and A. Derode, Waves Random Media **10**, R31 (2000).
[2] O. I. Lobkis and R. L. Weaver, J. Acoust. Soc. Am. **110**, 3011 (2001).
[3] Y. Lee, *Statistical Theory of Communication* (Wiley, New York, 1960).
[4] J. Rickett and J. Claerbout, Stanford Exploration Project Report No. SEP–92, p. 83 (1996); http://sepwww.stanford.edu/public/docs/sep92/james1/paper_html/index.html
[5] T. L. Duvall, S. M. Jeffferies, J. W. Harvey, and M. A. Pomerantz, Nature (London) **362**, 430 (1993).
[6] R. L. Weaver and O. I. Lobkis, Phys. Rev. Lett. **87**, 134301 (2001).
[7] M. Campillo and A. Paul, Science **299**, 547 (2003).
[8] G. C. Papanicolaou, L. V. Ryzhik, and K. B. Keller, Bull. Seismol. Soc. Am. **86**, 1107 (1996).
[9] R. L. Weaver, J. Acoust. Soc. Am. **71**, 1608 (1982).
[10] R. Hennino, N. Trégourès, N. M. Shapiro, L. Margerin, M. Campillo, B. A. van Tiggelen, and R. L. Weaver, Phys. Rev. Lett. **86**, 3447 (2001).
[11] J. A. Scales and A. E. Malcolm, Phys. Rev. E **67**, 046618 (2003).
[12] R. L. Weaver, J. Acoust. Soc. Am. **78**, 131 (1985).
[13] J. Scales and K. Van Wijk, Appl. Phys. Lett. **79**, 2294 (2001).
[14] A. Derode, E. Larose, M. Campillo, and M. Fink, Appl. Phys. Lett. **83**, 3054 (2003).
[15] A. Lagendijk and B. van Tiggelen, Phys. Rep. **270**, 143 (1996).
[16] M. van Rossum and T. Nieuwenhuizen, Rev. Mod. Phys. **71**, 313 (1999).
[17] N. Trégourès and B. van Tiggelen, Phys. Rev. E **66**, 36601 (2002).
[18] A. Derode, A. Tourin, and M. Fink, J. Appl. Phys. **85**, 6343 (1999).
[19] An n-length segment of the data gives the GF for the $2n+1$ time points centered at time zero.