Outline

• Introduction
  ▶ Why we image the Earth
  ▶ How data are collected
  ▶ Imaging vs inversion
  ▶ Underlying physical model

• Data Model

• Imaging methods
  ▶ Kirchhoff
  ▶ One-way methods
  ▶ Reverse-time migration
  ▶ Full-waveform inversion

• Comparison of Methods
Approximate Techniques

• Kirchhoff
  ▶ Integral technique
  ▶ Related to X-ray CT imaging
  ▶ Generalized Radon Transform
  ▶ Conventionally uses ray theory

• One-way
  ▶ Based on a paraxial approximation
  ▶ Usually computed with finite differences
One-Way Methods
Physical Motivation

- downward continuation
- imaging condition

Claerbout 71, 85
One-Way Methods
Physical Motivation

- downward continuation
- imaging condition

\[ \frac{v}{2} \]
One-Way Methods
Physical Motivation

- downward continuation
- imaging condition

Claerbout 71, 85
One-Way Methods

Physical Motivation

- downward continuation
- imaging condition

\[
\begin{align*}
V(x) \\
V(x)
\end{align*}
\]
\[ \delta G(s, r, t) = \int_X \int_T G_0(r, t-t_0, x)V(x) \partial_t^2 G_0(x, t_0, s) \, dx \, dt_0 \]

\[ \delta G(s, r, \omega) = -\int_X \omega^2 G_0(r, \omega, x)V(x)G_0(x, \omega, s) \, dx \]
A Data Model

$$\delta G(s, r, t) = \int_X \int_T G_0(r, t-t_0, x)V(x)\partial_t^2 G_0(x, t_0, s)dxdt_0$$

$$\delta G(s, r, \omega) = -\int_X \omega^2 G_0(r, \omega, x)V(x)G_0(x, \omega, s)dx$$

⇓

$$\int_X \omega^2 G_-(r, \omega, x)G_-(s, \omega, x)\tilde{V}(x)dx$$
One-Way Methods
Approximating the Wave Equation

Idea (1D, c constant):

\[(\partial_x^2 - \partial_t^2)u = (\partial_x - \partial_t)(\partial_x + \partial_t)u\]

c not constant:

\[(c(x)^2 \partial_x^2 - \partial_t^2)u = (c(x)\partial_x - \partial_t)(c(x)\partial_x + \partial_t)u \]
\[\quad - c(x)(\partial_x c(x)) \partial_x u\]

c(x) smooth \Rightarrow \text{better approximation}
One-Way Methods
Approximating the Wave Equation
Taylor (81), Stolk & de Hoop (05)

Multi-dimensional:
\[ Lu = f \]

\[
\partial_z \begin{pmatrix} u \\ \partial_z u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -A & 0 \end{pmatrix} \begin{pmatrix} u \\ \partial_z u \end{pmatrix} + \begin{pmatrix} 0 \\ -f \end{pmatrix}
\]

\[
A(z, x, \partial_x, \partial_t) = \partial_x^2 - c(z, x)^{-2} \partial_t^2
\]
Diagonalize in smooth background

\[ \partial_z \begin{pmatrix} u_+ \\ u_- \end{pmatrix} = \begin{pmatrix} iB_+ & 0 \\ 0 & iB_- \end{pmatrix} \begin{pmatrix} u_+ \\ u_- \end{pmatrix} + \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \]

\[ b_\pm(\xi, x, \omega) = \pm \sqrt{\xi^2 - c_0(x)^{-2}\omega^2} \]

\[ B_\pm \text{ FIOs} \]
Diagonalize in smooth background

$$\partial_z \begin{pmatrix} u_+ \\ u_- \end{pmatrix} = \begin{pmatrix} iB_+ & 0 \\ 0 & iB_- \end{pmatrix} \begin{pmatrix} u_+ \\ u_- \end{pmatrix} + \begin{pmatrix} f_+ \\ f_- \end{pmatrix}$$

Solution:

$$G_0 = \begin{pmatrix} G_+ & 0 \\ 0 & G_- \end{pmatrix}$$

$G_0$ propagator in $c_0$
One-Way Methods
Approximating the Wave Equation
Taylor (81), Stolk & de Hoop (05)

Relate $u_\pm$ to $u$ and $\partial_z u$

\[
\begin{pmatrix}
  u_+ \\
  u_-
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
  (Q_+)^{-1} & -\mathcal{H}Q_+ \\
  (Q_-)^{-1} & \mathcal{H}Q_-
\end{pmatrix} \begin{pmatrix}
  u \\
  \partial_z u
\end{pmatrix}
\]

\[
q_\pm = \left[ \left( \frac{\omega}{c(z, x)} \right)^2 - \|\xi\|^2 \right]^{-1/4}
\]

$Q_\pm$ differ in sub-principal parts
$Q_\pm \psi\text{DOs}$
\[ \delta G(s, r, \omega) \approx \int_{x} \omega^2 Q^*_- (r) G_- (r, \omega, x) V(x) G_+ (x, \omega, s) Q_+ (s) dx \, dz \]

\[ V(x) = \frac{1}{4} Q_- (z, x) \delta c(z, x) c_0(z, x)^{-3} Q^*_+ (z, x) \]
One-Way Methods
Modeling Data Stolk & de Hoop (05)

\[\delta G(s, r, \omega) \approx \int_{x} \omega^2 Q^* (r) G_-(r, \omega, x)V(x)G_+(x, \omega, s)Q_+(s)dx\,dz\]

\[V(x) = \frac{1}{4} Q_-(z, x) \delta c(z, x)c_0(z, x)^{-3}Q^*_+(z, x)\]

Reciprocity:
\[ \delta G(s, r, \omega) \approx \int_{x} \omega^2 Q^*_-(r)G_-(r, \omega, x)V(x)G_+(x, \omega, s)Q_+(s)dxdz \]

\[ V(x) = \frac{1}{4} Q_-(z, x)\delta c(z, x)c_0(z, x)^{-3}Q^*_+(z, x) \]

Drop Qs

\[ \int_{x} \int_{z} \omega^2 G_-(0, r, \omega, z, x)G_-(0, s, \omega, z, x) \]

\[ (E_2E_1a)(z, x)dzdxdx \]

\[ E_2 : b(z, r, s) \leftrightarrow \delta(t)b(z, r, s) \]

\[ E_1 : a(z, x) \leftrightarrow \delta(r - s)a(z, \frac{r+s}{2}) \]
One-Way Methods

Imaging  Claerbout (78) Stolk & de Hoop (06)

- downward continuation
- imaging condition
• downward continuation

\[ \hat{d}(z, s, r, t) = H(z, 0)^* Q^{-1} d(s, r, t) \]

\[
(H(z, z_0))(s, r, t, s_0, r_0) = (G_-(z, z_0))(r, t, r_0) \ast (G_-(z, z_0))(s, t, s_0)
\]

H is an FIO \( H \ast H \) is \( \Psi DO \)

• imaging condition
One-Way Methods

Imaging Claerbout (78) Stolk & de Hoop (06)

- downward continuation
- imaging condition

\[ V(z, x) \approx \hat{d}(z, x, x, 0) \]
\[ V(x) = \frac{1}{4} Q_-(z) \delta c(z, x)c_0(z, x)^{-3} Q_+(z) \]

Recall Q are $\Psi$DOs & $H^*H$ is $\Psi$DO

We have again located the singularities
One-Way Methods

Example
Approximate Techniques

- **Kirchhoff**
  - Integral technique
  - Related to X-ray CT imaging
  - Generalized Radon Transform
  - Conventionally uses ray theory

- **One-way**
  - Based on a paraxial approximation
  - Usually computed with finite differences
‘Exact’ Techniques

- **Reverse-time migration (RTM)**
  - Run wave-equation backward
  - Usually computed with finite differences
  - “No” approximations (to the acoustic, isotropic, linearized wave-equation, for smooth media assuming single scattering)

- **Full-waveform inversion (FWI)**
  - Iterative method to match the entire waveform
  - Gives smooth part of velocity model
\[ \delta G(s, r, t) = \int_X \int_T G_0(r, t-t_0, x)V(x) \partial_t^2 G_0(x, t_0, s) dx dt_0 \]

\[ \delta G(s, r, \omega) = - \int_X \omega^2 G_0(r, \omega, x)V(x)G_0(x, \omega, s) dx \]
\[
\delta G(s, r, t) = \int_X \int_T G_0(r, t-t_0, x)V(x) \partial_t^2 G_0(x, t_0, s)dxdt_0
\]

\[
\delta G(s, r, \omega) = - \int_X \omega^2 G_0(r, \omega, x)V(x)G_0(x, \omega, s)dx
\]

\[
F : \delta c \mapsto \delta G \text{ is again an FIO}
\]
Reverse-Time Migration

Forming an image

\[ \delta G(s, r, \omega) = - \int \omega^2 G_0(r, \omega, x)V(x)G_0(x, \omega, s)dx \]

\[ F : \delta c \mapsto \delta G \quad \text{is again an FIO} \]
\[ F^* : \delta G \mapsto \delta c \quad \text{is again an FIO} \]
\[ F^*F \text{ is } \Psi DO \text{ for complete coverage} \]

Stolk et al. (09)

- no direct rays (transversal intersection)
- no source-side caustics
- \( c_0 \in C^\infty \)
Reverse-Time Migration
Forming an Image

Procedure:
Whitmore (83), Loewenthal & Mufti (83), Baysal et al (83)

- back propagate in time
- imaging condition

\[ G_0(r, t, x) \]
\[ V(x) \]
Reverse-Time Migration
an Adjoint State Method

Lailly (83,84), Tarantola (84,86,87) Symes (09)

For a fixed source, $s$,

$$(c^{-2}\partial_t^2 - \nabla^2)q(x, t; s) = \int_{R_s} \delta G(r, t; s)\delta(x - r)dr$$

$q(\cdot, t, \cdot) = 0$ for $t > T$

receivers act as sources, reversed in time

$$\text{Im}(x) = \frac{2}{c^2(x)} \int \int q(x, t; s)\partial_t^2 G_0(x, t, s)dtds$$
Reverse-Time Migration
Example Liu et al (07)
Reverse-Time Migration
Example Liu et al (07)
Reverse-Time Migration
Example Liu et al (07)
‘Exact’ Techniques

- **Reverse-time migration (RTM)**
  - Run wave-equation backward
  - Usually computed with finite differences
  - “No” approximations (to the acoustic, isotropic, linearized wave-equation, for smooth media assuming single scattering)

- **Full-waveform inversion (FWI)**
  - Iterative method to match the entire waveform
  - Gives **smooth** part of velocity model
Recall our initial formulation:

\[ Lu := (\nabla^2 - \frac{1}{c^2} \partial^2_t)u = f \]

\[ u = 0 \quad t < 0 \]

\[ \partial_z u \big|_{z=0} = 0 \]

FWI attempts to solve for \( c \) directly given \( u,f \)

no splitting of \( c \)

but band limited data \( \Rightarrow \) smooth solution
Recall our initial formulation:

\[ Lu := (\nabla^2 - \frac{1}{c^2} \partial_t^2)u = f \]

Compute:

\[
\min_c \| Lu - d \|_{L^2((S,R) \times [0,T])}
\]
Compute:

\[ \min_c \| Lu - d \|_{L^2(S,R \times [0,T])} \]

- Objective function appears non-convex
  Can we restrict the domain of models so that it is? e.g. \( c_{\min} < c < c_{\max} \)
- What space must \( c \) be in for objective function to be differentiable?
- Data are finite bandwidth – we cannot resolve structures on all scales
Idea: Extend $\delta c(x)$ from $X$ to $\hat{X}$

Example: $\delta c(x) \leftrightarrow \delta \hat{c}(x, h) = \delta(h) \delta c(x)$

Claerbout (85) suggested this extension
Idea: Extend $\delta c(x)$ from $X$ to $\hat{X}$

Example: $\delta c(x) \mapsto \delta \hat{c}(x, h) = \delta(h) \delta c(x)$

Now extend

$$F[\delta c] \mapsto \delta G$$

to

$$\hat{F}[\delta \hat{c}(x, h)] \mapsto \delta G$$

s.t. $\hat{F}$ is ‘invertible’ (I − $\hat{F}^\dagger \hat{F}$ is smoothing)

Find $c_0$ s.t. $\text{supp} (\delta \hat{c}(x, h)) \subset \{(x, h)|h = 0\}$

Stolk & de Hoop (2001), Stolk et al (2005) show invertibility
Full-waveform Inversion
Work-around 2: Separation of Scales
Pratt (99), Virieux (09)

\[
\min_c \| Lu - d \|_{L^2((S,R) \times [0,T])} \text{ not convex}
\]

\[
\Rightarrow \| c_{\text{initial}} - c_{\text{true}} \| \text{ must be small}
\]

Solve for \( G(\omega) \) from \( \omega_{\text{min}} \) to \( \omega_{\text{max}} \)

No proof optimization converges
Full-waveform Inversion
Work-around 2: Separation of Scales, example
Virieux (09)
Outline

• Introduction
  ▶ Why we image the Earth
  ▶ How data are collected
  ▶ Imaging vs inversion
  ▶ Underlying physical model

• Data Model

• Imaging methods
  ▶ Kirchhoff
  ▶ One-way methods
  ▶ Reverse-time migration
  ▶ Full-waveform inversion

• Comparison of Methods
Figure 5  Kirchhoff migration image obtained by migrating all common-shot gathers for the model in Figure 1. Solid line shows boundary of salt body in Figure 1.
Figure 6  Phase-shift migration image obtained by migrating all common-shot gathers for the model in Figure 1. Solid line shows boundary of salt body in Figure 1.
Comparison of Methods
Kirchhoff vs One-way  Fehler & Huang (02)

Figure 7  Split-step Fourier migration image obtained by migrating all common-shot gathers for the model in Figure 1. Solid line shows boundary of salt body in Figure 1.

Split-step
Comparison of Methods
Kirchhoff vs RTM Zhu & Lines (98)

Fig. 1. Kirchhoff migration impulses. The migration is based on a blocked velocity model with a normal fault developed throughout the depth range. The velocity in each block linearly increases with depth.

Fig. 2. Reverse-time migration impulses. The migration is based on the same input data and the same velocity model as used in Figure 1.
Fig. 5. The final migration section of the Marmousi model data set produced by the Kirchhoff integral method.
Comparison of Methods
Kirchhoff vs RTM Zhu & Lines (98)

Fig. 6. The final migration section of the Marmousi model data set produced by the reverse-time migration.
Comparison of Methods
One-Way vs RTM Farmer (06)
Comparison of Methods
One-Way vs RTM Farmer (06)

One-way
Comparison of Methods
One-Way vs RTM Farmer (06)

Reverse-time
Comparison of Methods
One-Way vs RTM Farmer (06)

One-way
Comparison of Methods
One-Way vs RTM Farmer (06)

Reverse-time
Comparison of Methods
Real Data Farmer 2006

Kirchhoff
Comparison of Methods
Real Data Farmer 2006

One-way
Comparison of Methods
Real Data Farmer 2006

Reverse-time
Summary of Methods

- Kirchhoff requires a lot of rays (or a simple model)
- One-way doesn’t handle turning waves
- RTM separation of up/down-going waves is challenging
  \[ c_0 \text{ is key} \]
- FWI optimization doesn’t converge from an arbitrary starting model
  
  ... and we never spoke about the source ...
  
  ... or multiple-scattering ...
Edip Baysal, Dan D. Kosloff, and John W. C. Sherwood.  
Reverse time migration.  

J. F. Claerbout.  
Imaging the Earth’s Interior.  

Jon F. Claerbout.  
Toward a unified theory of reflector mapping.  

P. A. Farmer.  
Reverse time migration: Pushing beyond wave equation.  

Michael C. Fehler and Lianjie Huang.  
Modern imaging using seismic reflection data.  

P. Lailly.  
The Seismic Inverse Problem as a Sequence of Before-Stack Migrations, pages 206–220.  

P. Lailly.  
Migration Methods: Partial but Efficient Solutions to the Seismic Inverse Problem.  

Faqi Liu, Guanquan Zhang, Scott A. Morton, and Jacques P. Leveille.  
Reverse-time migration using one-way wavefield imaging condition.  

Dan Loewenthal and Irshad R. Mufti.  
Reversed time migration in spatial frequency domain.  
C. C. Stolk and M. V. de Hoop.
Modeling of seismic data in the downward continuation approach.

C. C. Stolk and M. V. de Hoop.
Seismic inverse scattering in the downward continuation approach.

Christiaan C. Stolk, Maarten V. de Hoop, and William W. Symes.
Kinematics of shot-geophone migration.

W. W. Symes.
The seismic reflection inverse problem.

A. Tarantola.
Inverse Problem Theory.

Albert Tarantola.
Inversion of seismic reflection data in the acoustic approximation.

Albert Tarantola.
A strategy for nonlinear elastic inversion of seismic reflection data.

M. E. Taylor.
Pseudodifferential Operators.

J. Virieux and S. Operto.
An overview of full-waveform inversion in exploration geophysics.
N. D. Whitmore.
Iterative depth migration by backward time propagation.

Jinming Zhu and Larry R. Lines.
Comparison of kirchhoff and reverse-time migration methods with applications to prestack depth imaging of complex structures.