# A Potential Method for Three-dimensional Numerical Modeling of Geophysical Electromagnetic Problems

SeyedMasoud Ansari
Colin G. Farquharson

18 June, 2014

# Outline

# Maxwell's equations

Deriving a PDE from Maxwell's equations, constitutive relations, Ohm's law in the quasi-static regime.

Faraday's law of induction

$$\nabla \times \mathbf{E} + i\omega\mu_0\mathbf{H} = \mathbf{0}, \tag{1}$$

Ampère's law

$$\nabla \times \mathbf{H} - \mathbf{J} = \mathbf{J}^s, \tag{2}$$

Ohm's law

$$\mathbf{J} = \sigma\mathbf{E} \tag{3}$$

$\mathbf{E}(\mathbf{r},\omega)$ : Electric field, $\mathbf{H}(\mathbf{r},\omega)$ : Magnetic field intensity, $\mathbf{J}^s(\mathbf{r},\omega)$ : source current density, and $\sigma(r)$ : Electrical conductivity

# E-field PDE and introduction to potentials

$$\nabla \times \nabla \times \mathbf{E} + i\omega\mu_0\sigma\mathbf{E} = -i\omega\mu_0\mathbf{J}^s. \tag{4}$$

Because of the vanishing conductivity term, $i\omega\mu_0\sigma\mathbf{E}$ for lower frequencies it is difficult to solve the E-field equation.

High condition number for the system: Total-field, Finite-Element solution on Unstructured meshes → Iterative solution very slow.

- Decomposition of the electric field into potentials

$$\mathbf{E} = -i\omega\mathbf{A} - \nabla\phi \tag{5}$$

$\mathbf{A}(\mathbf{r}, \omega)$ and $\phi(\mathbf{r}, \omega)$ are vector and scalar potentials respectively.

$$\nabla \times \nabla \times \mathbf{A} + i\omega\mu_0\sigma\mathbf{A} + \mu_0\sigma\nabla\phi = \mu_0\mathbf{J}^s. \tag{6}$$

# Equation of Conservation of Charge

In order to solve for $\mathbf{A}$ and $\phi$ a second equation is required.

$$\nabla \cdot (\sigma \mathbf{E}) = \begin{cases} -\nabla \cdot \mathbf{J}^s & \text{source location,} \\ 0 & \text{otherwise.} \end{cases} \quad (7)$$

For low frequencies also the coupling between $\mathbf{E}$ and $\mathbf{H}$ in the Maxwell's equations reduces and electric charges become important in distorting the field in the conductive medium.

Conservation of charge in the decomposed form

$$-i\omega \nabla \cdot (\sigma \mathbf{A}) - \nabla \cdot (\sigma \nabla \phi) = -\nabla \cdot \mathbf{J}^s. \quad (8)$$

# Reiterating the Equations

The E-field system

$$\nabla \times \nabla \times \mathbf{E} \; + \; i\omega\mu_0\sigma\mathbf{E} \; = \; -i\omega\mu_0\mathbf{J}^s. \tag{9}$$

---

The Decomposed system

$$\nabla \times \nabla \times \mathbf{A} \; + \; i\omega\mu_0\sigma\mathbf{A} \; + \; \mu_0\sigma\nabla\phi \; = \; \mu_0\mathbf{J}^s, \tag{10}$$

$$-i\omega\nabla\cdot(\sigma\mathbf{A}) \; - \; \nabla\cdot(\sigma\nabla\phi) \; = \; -\nabla\cdot\mathbf{J}^s. \tag{11}$$

# Introduction to discretization

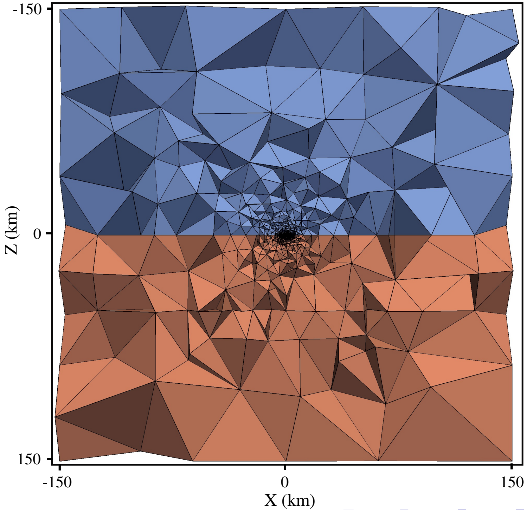
Unstructured Tetrahedral meshes

Mesh generator tool: TetGen (Si, 2007)

Mesh accuracy requirements:

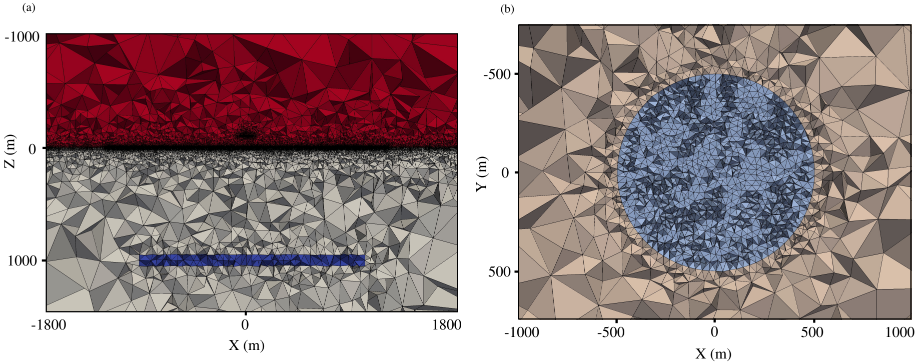
- Largest to smallest cell size
- Radius-Edge Ratio
- Dihedral angles for each cell

ParaView vertical section →



# Working with unstructured meshes

Flexibility to optimally generate curvilinear contacts with minimal staircasing effects




Only refining the mesh at the region of interest

# The method of weighted residuals

Finds the best approximation for $\mathbf{A}$ and $\phi$ by reducing the residual to its minimum value.

$$\mathrm{R} = \int_{\Omega} \mathbf{W} \cdot \mathbf{r} \; \mathrm{d}\Omega \; = \; 0, \tag{12}$$

with

$$\mathbf{r} = \nabla \times \nabla \times \tilde{\mathbf{A}} \; + \; i\omega\mu_0\sigma\tilde{\mathbf{A}} \; + \; \mu_0\sigma\nabla\tilde{\phi} \; - \mu_0\mathbf{J}^s.$$

$$\rho = \int_{\Omega} \; \mathrm{v} \; \mathrm{r} \; \mathrm{d}\Omega \; = \; 0, \tag{13}$$

with

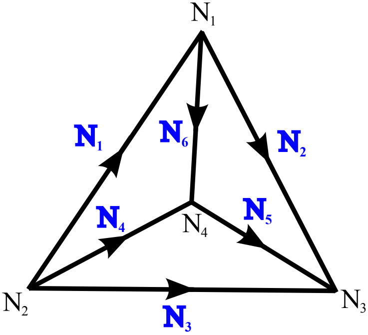
$$\mathrm{r} = -\mathrm{i}\omega\nabla \cdot (\sigma\tilde{\mathbf{A}}) \; - \; \nabla \cdot (\sigma\nabla\tilde{\phi}) \; + \nabla \cdot \mathbf{J}^{\mathrm{s}}.$$

# Finite-element basis functions

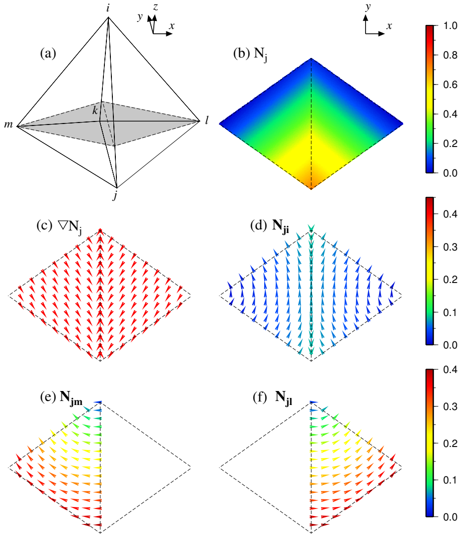
Vector basis functions or edge-elements for the approximate vector potential, $\tilde{\mathbf{A}}$
Scalar basis functions or nodal-elements for the approximate scalar potential, $\tilde{\phi}$

$$\tilde{\mathbf{A}} = \sum_{\mathrm{j}=1}^{\mathrm{n_A}} \tilde{\mathrm{A}}_{\mathrm{j}}\ \mathbf{N}_{\mathrm{j}}, \qquad (14)$$

$$\tilde{\phi} = \sum_{\mathrm{k}=1}^{\mathrm{n}_\phi} \tilde{\phi}_{\mathrm{k}}\ \mathrm{N}_{\mathrm{k}}. \qquad (15)$$



# Linear Basis functions



# Weighted PDEs

$$\int_\Omega (\nabla \times \mathbf{W}) \cdot (\nabla \times \tilde{\mathbf{A}}) \, \mathrm{d}\Omega \; - \; \int_{\gamma+\Gamma} \mathbf{W} \times (\nabla \times \tilde{\mathbf{A}}) \cdot \hat{\mathrm{n}} \, \mathrm{dS} \; + \\ \mathrm{i}\omega\mu_0 \int_\Omega \sigma \, \mathbf{W} \cdot \tilde{\mathbf{A}} \, \mathrm{d}\Omega \; + \; \mu_0 \int_\Omega \sigma \, \mathbf{W} \cdot \nabla\tilde{\phi} \, \mathrm{d}\Omega \; = \; \mu_0 \int_\Omega \mathbf{W} \cdot \mathbf{J}^{\mathrm{s}} \, \mathrm{d}\Omega. \tag{16}$$

$$\mathrm{i}\omega \int_\Omega \nabla \mathrm{v} \cdot \sigma\tilde{\mathbf{A}} \, \mathrm{d}\Omega \; - \; \mathrm{i}\omega \int_{\gamma+\Gamma} \mathrm{v} \, \sigma\tilde{\mathbf{A}} \cdot \hat{\mathbf{n}} \, \mathrm{dS} \; + \\ \int_\Omega \nabla \mathrm{v} \cdot \sigma\nabla\tilde{\phi} \, \mathrm{d}\Omega \; - \; \int_{\gamma+\Gamma} \mathrm{v} \, \sigma\nabla\tilde{\phi} \cdot \hat{\mathbf{n}} \, \mathrm{dS} \; = \; -\int_\Omega \mathrm{v} \, \nabla \cdot \mathbf{J}^{\mathrm{s}} \, \mathrm{d}\Omega. \tag{17}$$

Galerkin Method: $\mathbf{W} = \mathbf{N}$ and $\mathrm{v} = \mathrm{N}$

Dirichlet Boundary Conditions

# The $\mathbf{A} - \phi$ system in the matrix form

$$\begin{pmatrix} \mathbf{C} + \mathrm{i}\omega\mu_0\mathbf{D} & \mu_0\mathbf{F} \\ \mathrm{i}\omega\mathbf{G} & \mathbf{H} \end{pmatrix} \begin{pmatrix} \tilde{\mathrm{A}} \\ \tilde{\phi} \end{pmatrix} = \begin{pmatrix} \mu_0\mathrm{S}_1 \\ \mathrm{S}_2 \end{pmatrix} \tag{18}$$

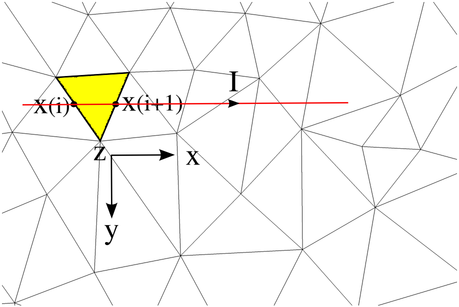
Finite-element inner product functionals:
$\mathbf{C} = \mathcal{F}(\mathbf{N})$
$\mathbf{D} = \mathcal{F}(\sigma, \mathbf{N})$
$\mathbf{F} = \mathcal{F}(\sigma, \mathbf{N}, \nabla\mathrm{N})$
$\mathbf{G} = \mathcal{F}(\sigma, \mathbf{N}, \nabla\mathrm{N})$
$\mathbf{H} = \mathcal{F}(\sigma, \nabla\mathrm{N})$.

$\tilde{\mathrm{A}}$ and $\tilde{\phi}$ are the approximated potentials.

# Source function

$$S_1 = \int_\Omega \mathbf{N}_i \cdot \mathbf{J}^s \, d\Omega \, , \qquad S_2 = -\int_\Omega N_l \, \nabla \cdot \mathbf{J}^s \, d\Omega.$$

Arbitrarily positioned in the mesh.




$$\mathbf{J} = I \, (\mathcal{H}(x_{i+1}) - \mathcal{H}(x_i)) \, \delta(y - y_0) \, \delta(z - z_0) \tag{19}$$

# Iterative solution and preconditioning

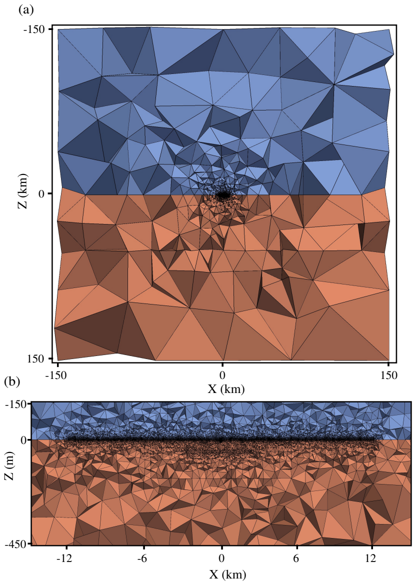
$$\begin{pmatrix} \mathbf{C} + \mathrm{i}\omega\mu_0\mathbf{D} & \mu_0\mathbf{F} \\ \mathrm{i}\omega\mathbf{G} & \mathbf{H} \end{pmatrix} \begin{pmatrix} \tilde{\mathrm{A}} \\ \tilde{\phi} \end{pmatrix} = \begin{pmatrix} \mu_0\mathrm{S}_1 \\ \mathrm{S}_2 \end{pmatrix} \tag{20}$$

ILUT preconditioning (Saad, 1990) with an appropriate fill-in factor prior to solution

Iterative solver of GMRES from SPARSKIT (Saad, 1990): A generalized minimum residual method in the Krylov subspace

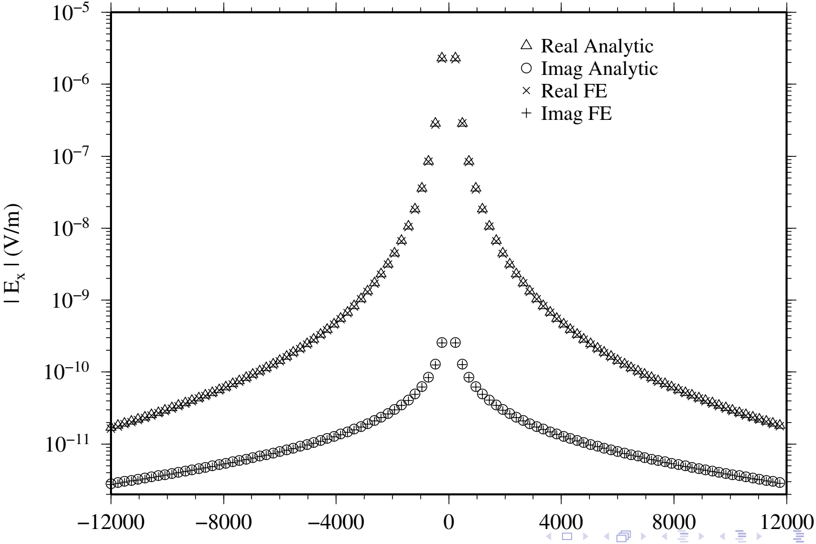
$$\tilde{\mathbf{E}} = -\mathrm{i}\omega\tilde{\mathbf{A}} - \nabla\tilde{\phi}$$

Frequency = 0.1 Hz, cells: 708796, nodes: 116058, edges: 825232. $\sigma_{\mathrm{air}} = 10^{-8}$ S/m, $\sigma_{\mathrm{Earth}} = 0.01$ S/m

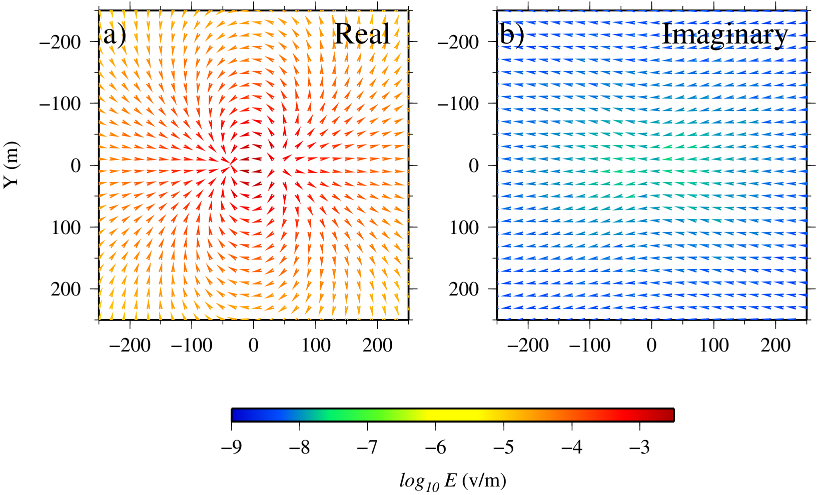


# Electric dipole source

Comparison with the analytic total field solution.

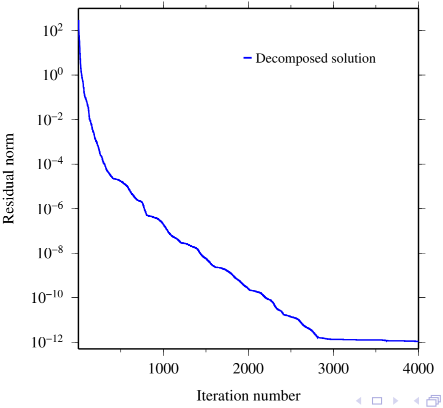


# Total electric field arrows in a horizontal plane z = 50 m



# Solver parameters

GMRES solver, ILUT preconditioner with lfil = 3.
Dimension of the Krylov subspace: 200.
Residual norms: $||Ax - b|| = 10^{-12}$ after 4000 iterations; Relative residual norm $\frac{||Ax-b||}{||b||} = 2.95 \times 10^{-10}$ for the final solution.
Computation time for the solution was roughly 30 minutes on a Apple Mac Pro computer (2.4 GHz Quad-core Intel Xenon processor) with a total memory usage of 8 Gbytes.



# $\mathbf{A} - \phi$ solution against the E-field solution

The E-field system

$$\nabla \times \nabla \times \mathbf{E} + i\omega\mu_0\sigma\mathbf{E} = -i\omega\mu_0\mathbf{J}^s,$$

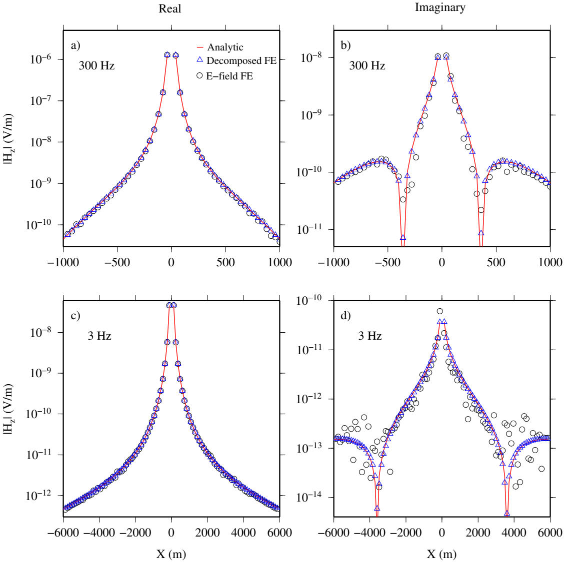
$$(\mathbf{C} + \mathrm{i}\omega\mu_0\mathbf{D})\,\tilde{\mathrm{E}} = \mathrm{i}\omega\mu_0\mathrm{S}_1 \tag{21}$$

---

The $\mathbf{A} - \phi$ system

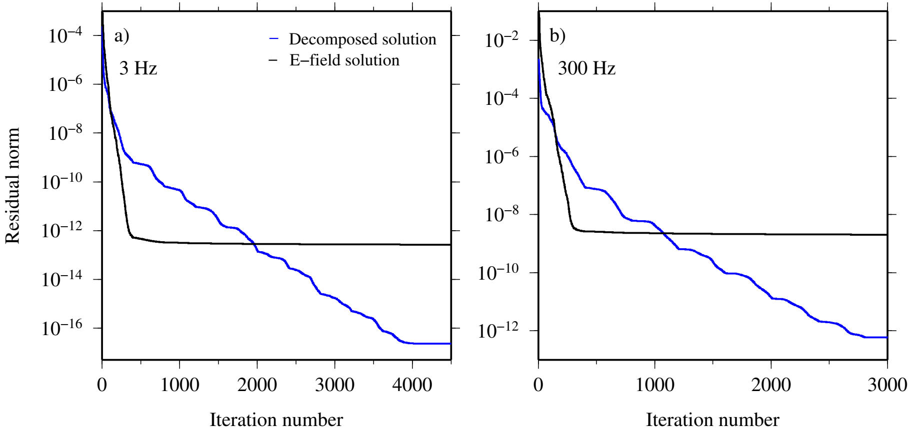
$$\begin{pmatrix} \mathbf{C} + \mathrm{i}\omega\mu_0\mathbf{D} & \mu_0\mathbf{F} \\ \mathrm{i}\omega\mathbf{G} & \mathbf{H} \end{pmatrix} \begin{pmatrix} \tilde{\mathrm{A}} \\ \tilde{\phi} \end{pmatrix} = \begin{pmatrix} \mu_0\mathrm{S}_1 \\ \mathrm{S}_2 \end{pmatrix} \tag{22}$$

Magentic dipole source; half-space model; 660491 cells, 107922, and 768795 edges; frequencies of 3 and 300 Hz; 8 Gbytes

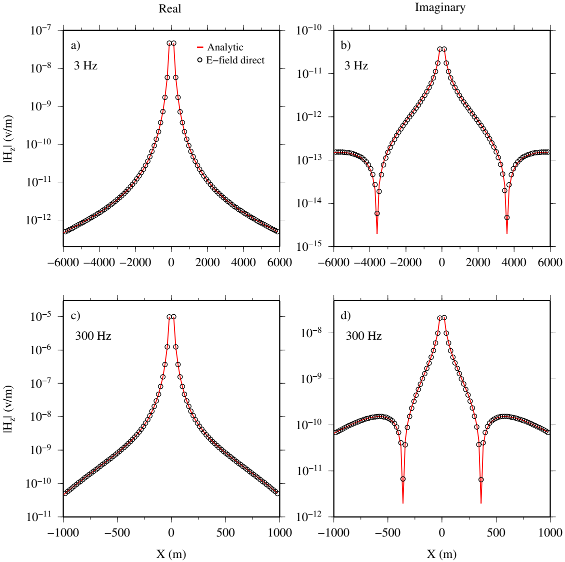


# Residual norms

Rapid convergence of the $\mathbf{A} - \phi$ solution
Slow convergence of the E-field solution

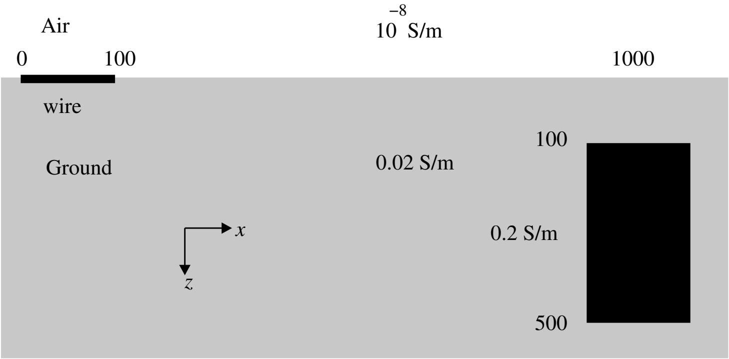


Direct solution for the E-field system; MUMPS solver is used
Average computation time and memory usage: 536 s and 19 Gbytes



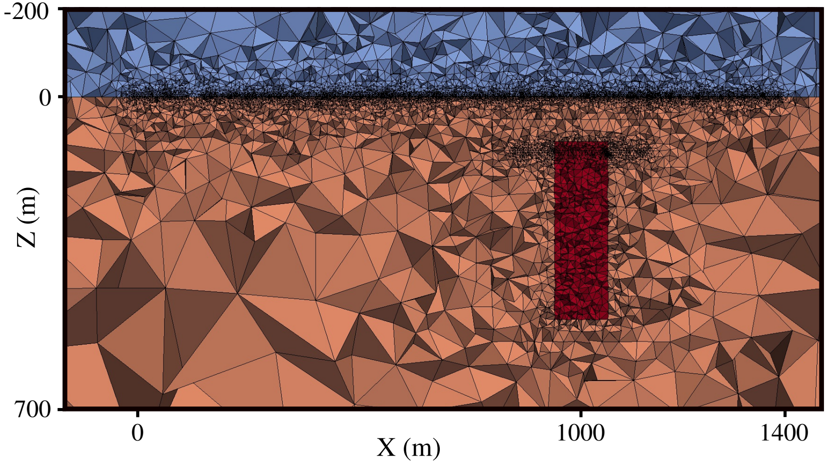
# Grounded wire and prism

A mineral exploration scenario designed by Li, Oldenburg and Shekhtman, 1999 : DCIP3D
Farquharson and Oldenburg, 2002 : Verification of the Integral Equation code



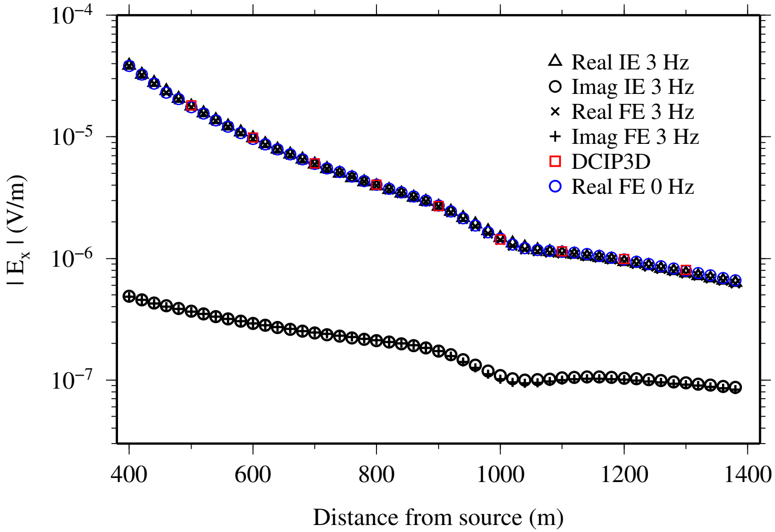
# Unstructured Mesh

cells: 613300, nodes: 99855, and edges: 713542
Frequency = 3 Hz

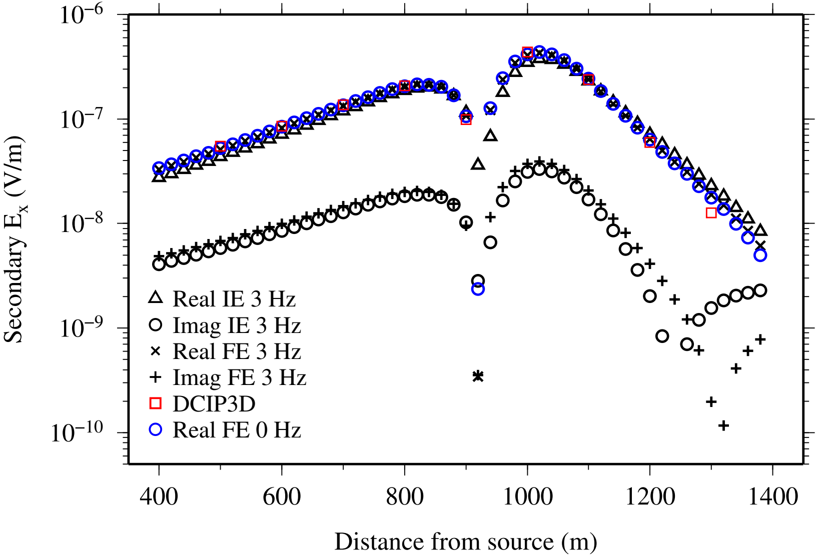


# Electric fields

Comparison against the Integral Equation and DC-resistivity.
The code works for f = 0 Hz.

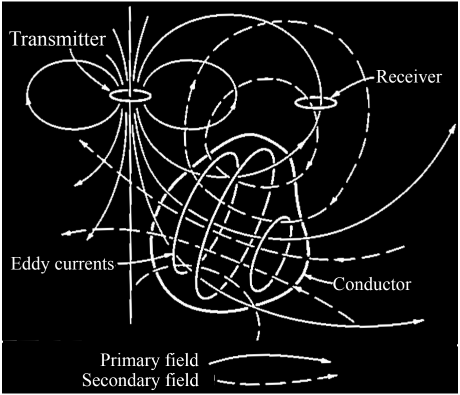


## Scattered fields

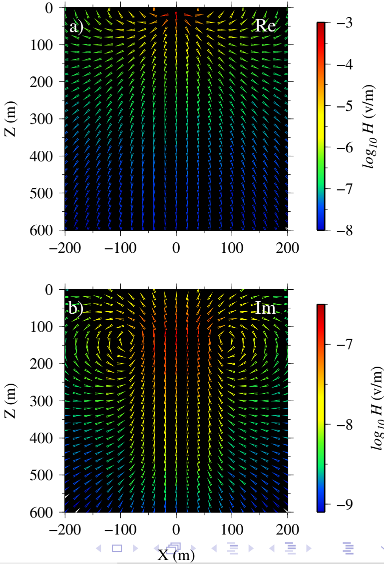


# Inductive concept

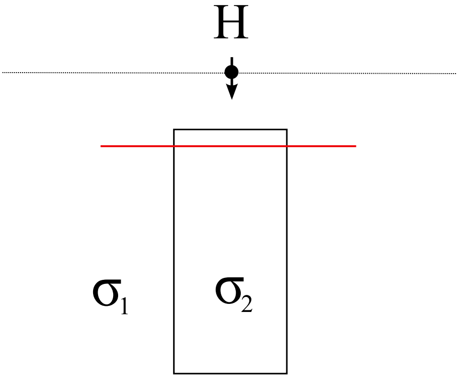
$$\tilde{\mathbf{E}} = -i\omega\tilde{\mathbf{A}} - \nabla\tilde{\phi}$$

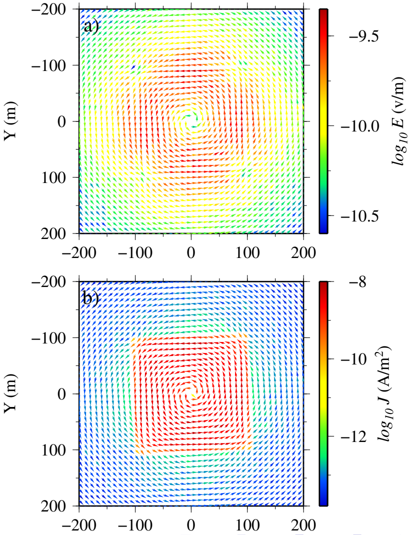



(Grant and West, 1965)

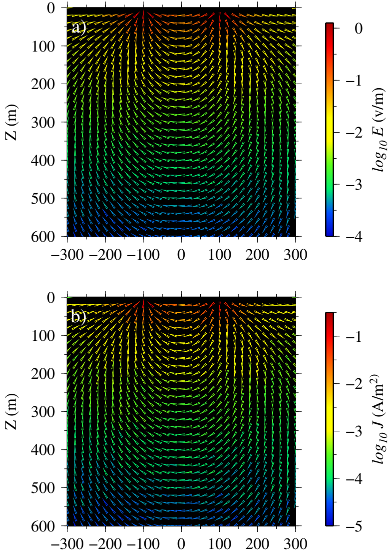


# Inductive concept cont.

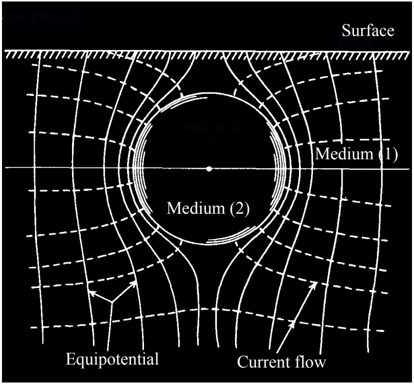


# Galvanic concept

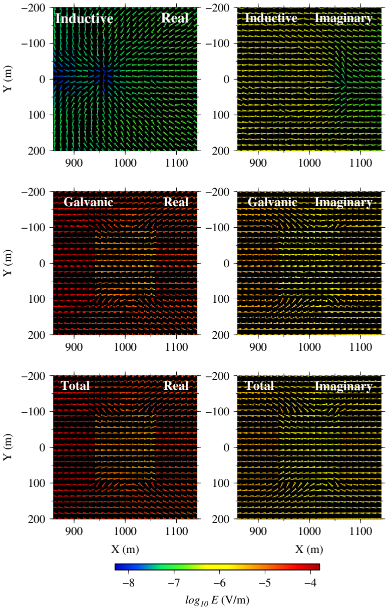



$$\tilde{\mathbf{E}} = -i\omega\tilde{\mathbf{A}} - \nabla\tilde{\phi}$$

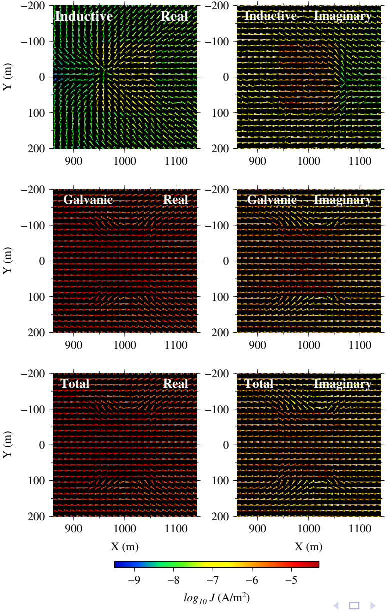



(Telford et al., 1990)

# Inductive and Galvanic fields at z = 120 m

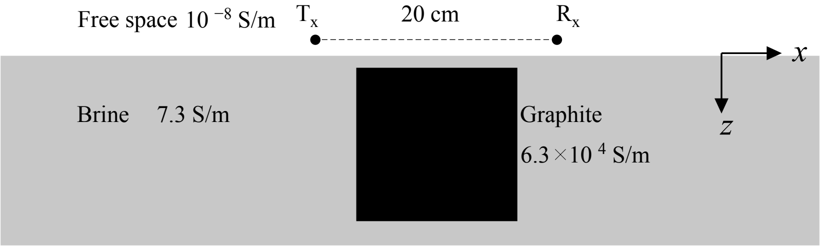


# Inductive and Galvanic current density at z = 120 m

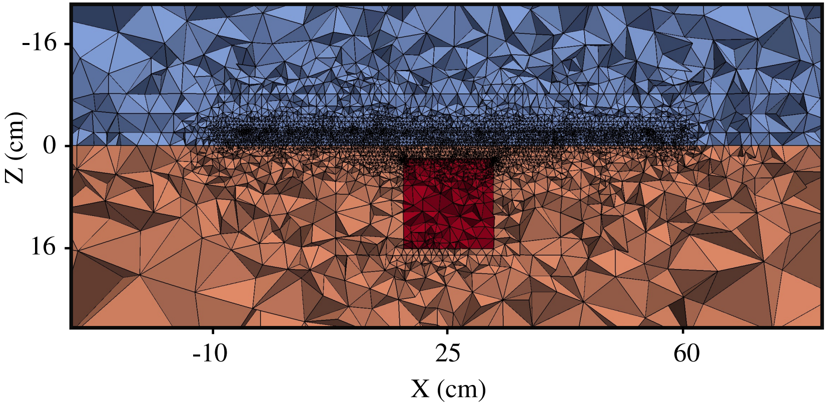


# Transmitter-Receiver pair and cube in brine

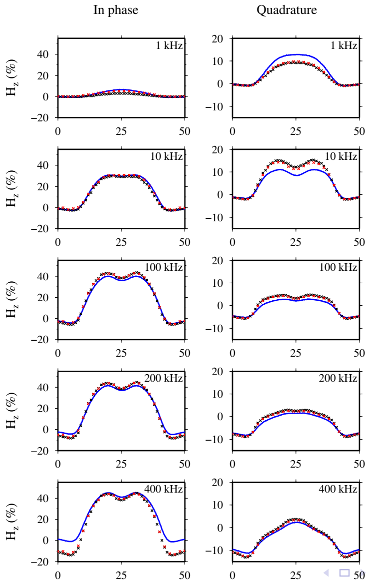
Five frequencies 1, 10, 100, 200, and 400 kHz.



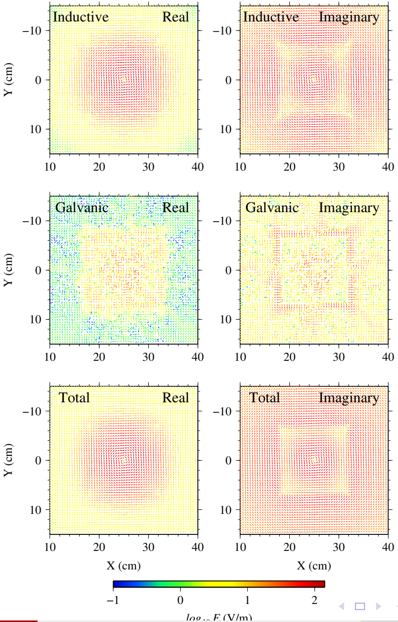
# Unstructured Mesh



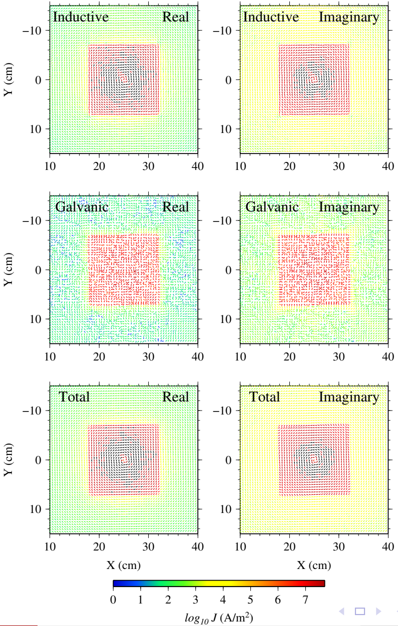
# Normalized magnetic fields: FE, FV of Jahandari and Farquharson, 2013, Physical Scale modeling of Farquharson et al., 2006



# Inductive and Galvanic fields, freq = 100 kHz

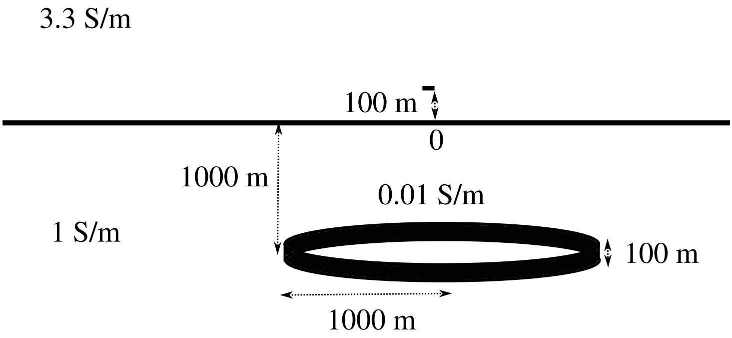


# Inductive and Galvanic current densities

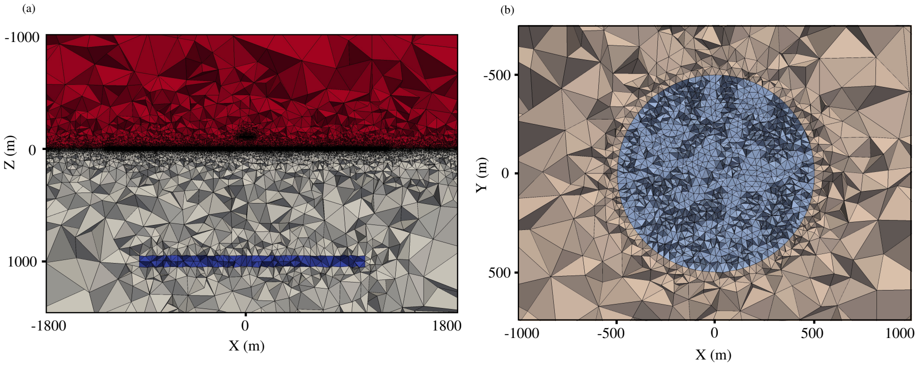


# Disk model in marine sediments

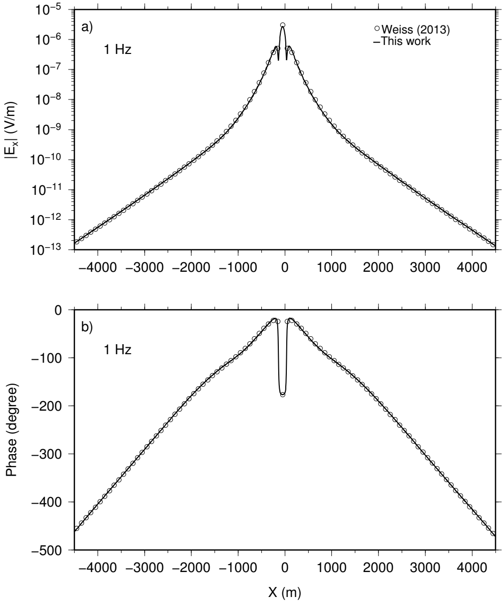
Frequency of 1 Hz



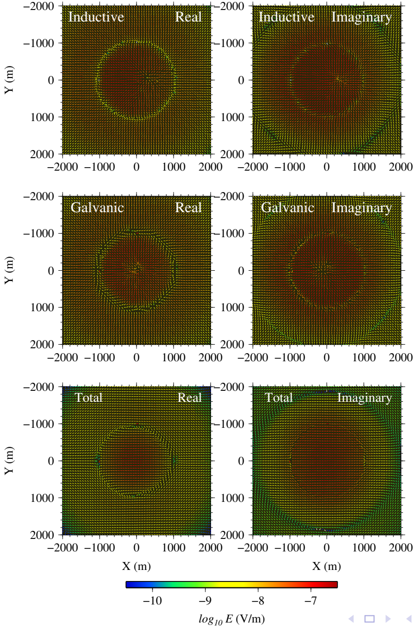
# Unstructured mesh



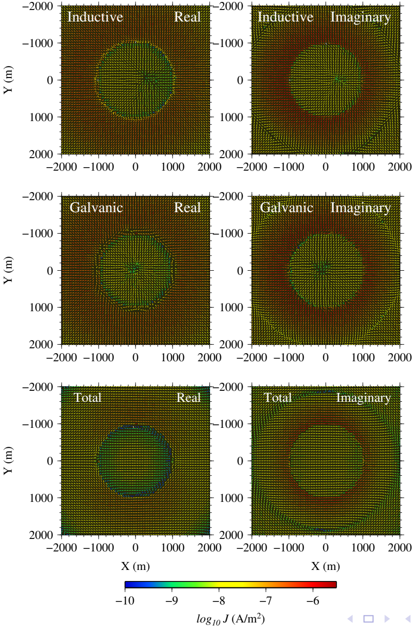
# In-line total electric fields and phase, Weiss Finite Volume approach (2013)



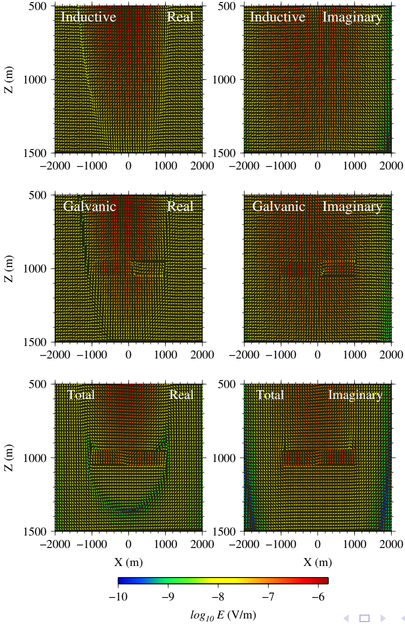
# Inductive and Galvanic fields: in the horizontal plane of z = 1000 m



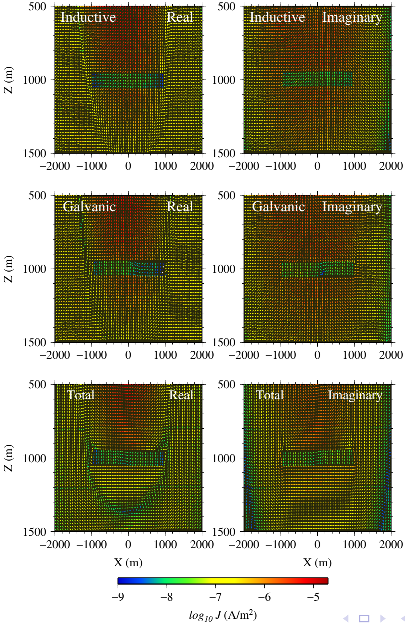
# Inductive and Galvanic current densities: in the horizontal plane of z = 1000 m



# Inductive and Galvanic fields: in the vertical plane of y = 0 m



# Inductive and Galvanic current densities: in the vertical plane of y = 0 m



# Uniqueness problem: Investigating the solutions of three systems

1. **Iterative solution to the un-gauged $\mathbf{A} - \phi$ system:** $\nabla \cdot \mathbf{A} = 0$ **is not enforced explicitly**

2. **Direct solution to the gauged $\mathbf{A} - \phi$ system:** $\nabla \cdot \mathbf{A} = 0$ **is enforced explicitly**

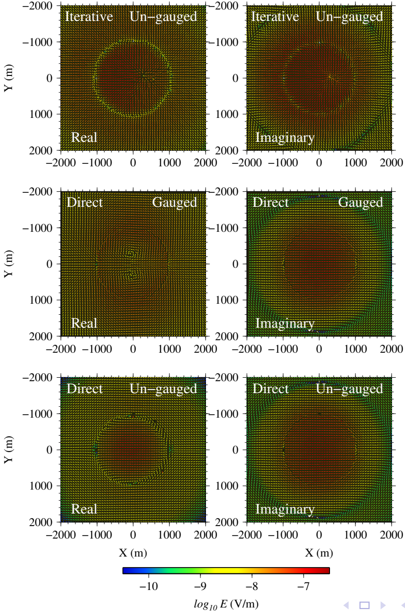
$$\nabla \times \nabla \times \mathbf{A} + \mathrm{i}\omega\mu_0\sigma\mathbf{A} + \mu_0\sigma\nabla\phi = \mu_0\mathbf{J}^{\mathrm{s}}, \quad (23)$$

$$-\nabla \cdot (\sigma\nabla\phi) = -\nabla \cdot \mathbf{J}^{\mathrm{s}}.$$

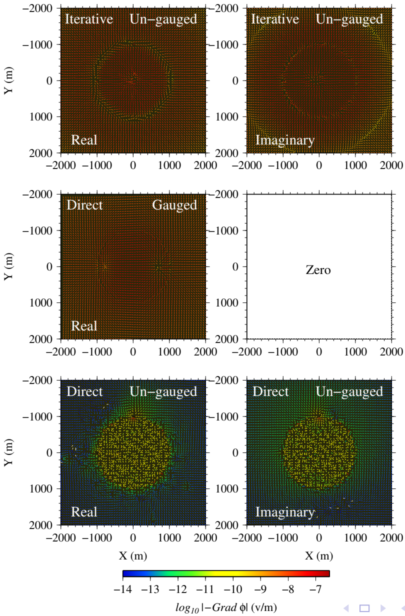
   Direct because iterative is slow

3. **Direct solution to the un-gauged $\mathbf{A} - \phi$ system:** $\nabla \cdot \mathbf{A} = 0$ **is not enforced explicitly**

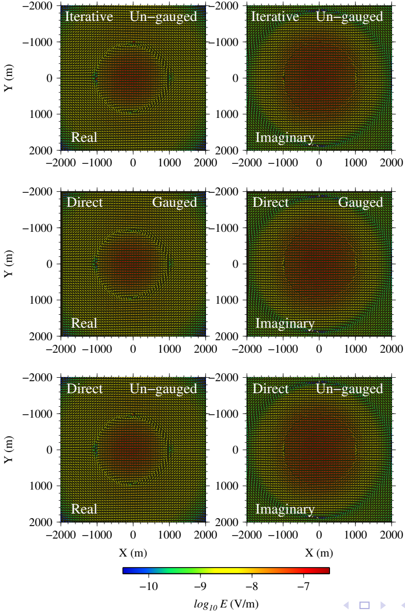
# Non-unique vector potentials or inductive parts: $-i\omega\mathbf{A}$



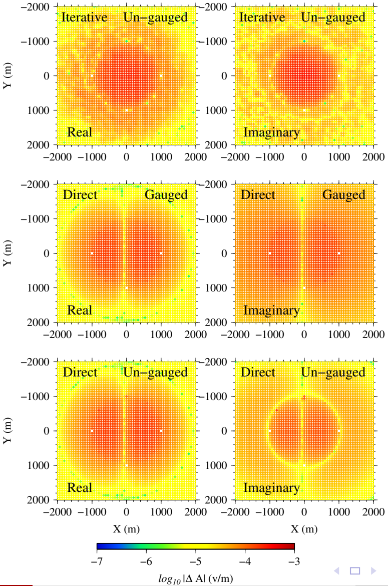
# Non-unique scalar potentials or galvanic parts: $-\nabla\phi$



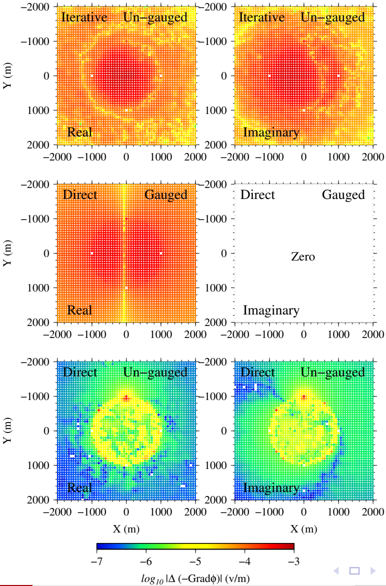
# Unique electric fields



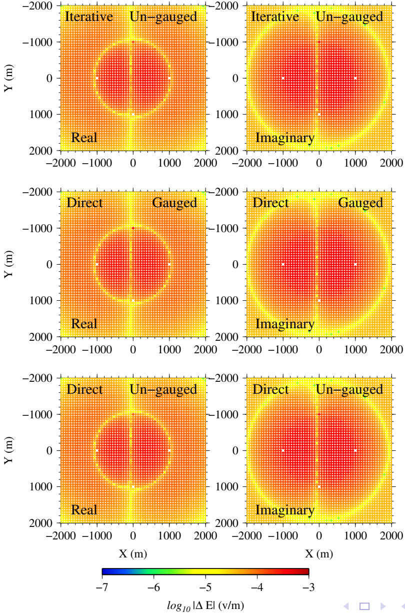
# Normal component of the inductive part across interface z = 950 m, $-i\omega\mathbf{A}$: Non-unique



# Normal component of the galvanic part across interface, $-\nabla\phi$: Non-unique



# Normal component of the field across interface - Unique electric fields



# Conclusions

- A 3D finite-element solution for forward modeling of geophysical electromagnetic problems is presented.
- The algorithm is written for the total field approximation on unstructured tetrahedral meshes.
- The approach is based on decomposing the electric field into vector and scalar potentials in the Helmholtz equation and equation of conservation of charge.
- The decomposition is done not only from the perspective of solving the equations efficiently, but also in order to delve into the physical meaning of the inductive and galvanic components.
- We verified the method for multiple examples in different geophysical scenarios where either electric and magnetic sources are used.