An algorithm for the three-dimensional inversion of magnetotelluric data

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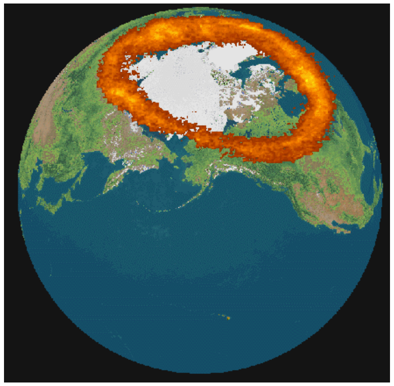
- ⋆ Randall Mackie, Yuji Mitsuhata.

Forward modelling: equations

⋆ Forward modelling for MT …

→ homogeneous equations:

$$
\begin{aligned}
\nabla \times \mathbf{E} \; - \; i\omega\mu_0\mathbf{H} \; &= \; 0, \\
\nabla \times \mathbf{H} \; - \; \mathbf{J} \; &= \; 0, \\
\nabla \cdot \mathbf{J} \; &= \; 0, \\
\mathbf{J} \; - \; \sigma\mathbf{E} \; &= \; 0;
\end{aligned}
$$



→ but *in*homogeneous boundary conditions.

Forward modelling: equations

⋆ Introduce vector and scalar potentials:

$$\mathbf{E} \;=\; \mathbf{A} \;+\; \nabla\phi,$$

and the Coulomb gauge condition:

$$\nabla \cdot \mathbf{A} \;=\; 0.$$

Hence ...

$$\begin{aligned} \nabla^2\mathbf{A} \;+\; i\omega\mu_0\sigma\big(\mathbf{A} + \nabla\phi\big) \;&=\; 0, \\ \nabla \cdot \big(\sigma\mathbf{A}\big) \;+\; \nabla \cdot \big(\sigma\nabla\phi\big) \;&=\; 0, \end{aligned}$$

that is,

$$\mathcal{A}(\mathbf{m})\,\mathbf{u} \;=\; 0,$$

with inhomogeneous boundary conditions on $\mathbf{A}$ & ϕ.

Forward modelling: equations

⋆ Or … a primary–secondary field separation:

$$\mathbf{E} \;=\; \mathbf{E}_p \;+\; \mathbf{E}_s \quad \& \quad \mathbf{H} \;=\; \mathbf{H}_p \;+\; \mathbf{H}_s.$$

And …

$$\mathbf{A} \;=\; \mathbf{A}_p \;+\; \mathbf{A}_s, \quad \& \quad \phi \;=\; \phi_p \;+\; \phi_s.$$

Hence …

$$\begin{aligned}\nabla^2\mathbf{A}_s \;+\; i\omega\mu_0\sigma\big(\mathbf{A}_s + \nabla\phi_s\big) \;&=\; -i\omega\mu_0\Delta\sigma\mathbf{E}_p,\\ \nabla\cdot\big(\sigma\mathbf{A}_s\big) \;+\; \nabla\cdot\big(\sigma\nabla\phi_s\big) \;&=\; -\nabla\cdot\big(\Delta\sigma\mathbf{E}_p\big),\end{aligned}$$

that is,

$$\mathcal{A}(\mathbf{m})\,\mathbf{u}_s \;=\; \hat{\mathbf{q}}(\mathbf{m}),$$

with homogeneous boundary conditions on $\mathbf{A}_s$ & ϕ_s.

Forward modelling: algorithm

- Rectangular mesh.

- The *scalar potential* is approximated by its values at *cell centres*; the *vector potential* is approximated by its normal components at the centres of *cell faces* (Haber, Ascher, Aruliah & Oldenburg, 2000; Haber & Ascher 2001).

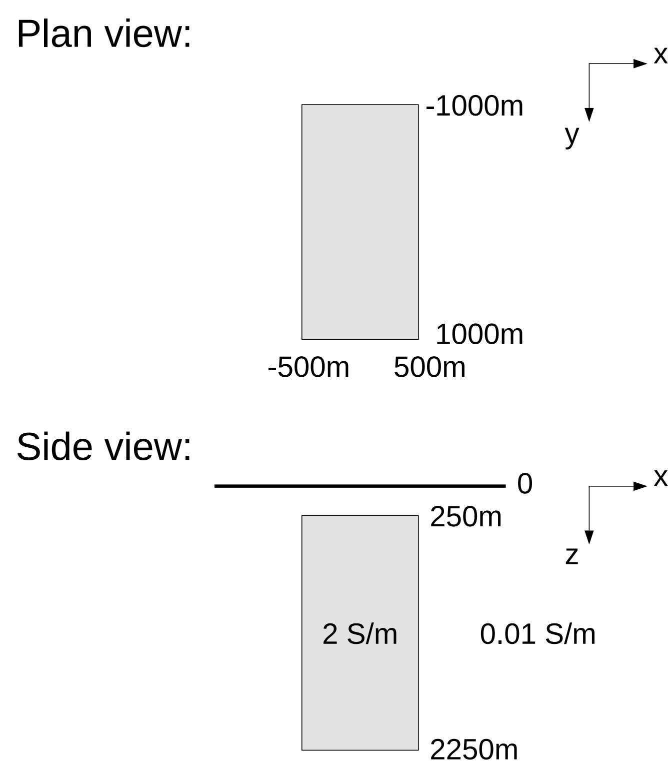
- A *finite volume* technique is used to obtain the system of equations: this naturally leads to *harmonic averaging* of conductivities in neighbouring cells.

- The system of equations is solved using a stabilised bi-conjugate gradient algorithm with ILU preconditioner †.

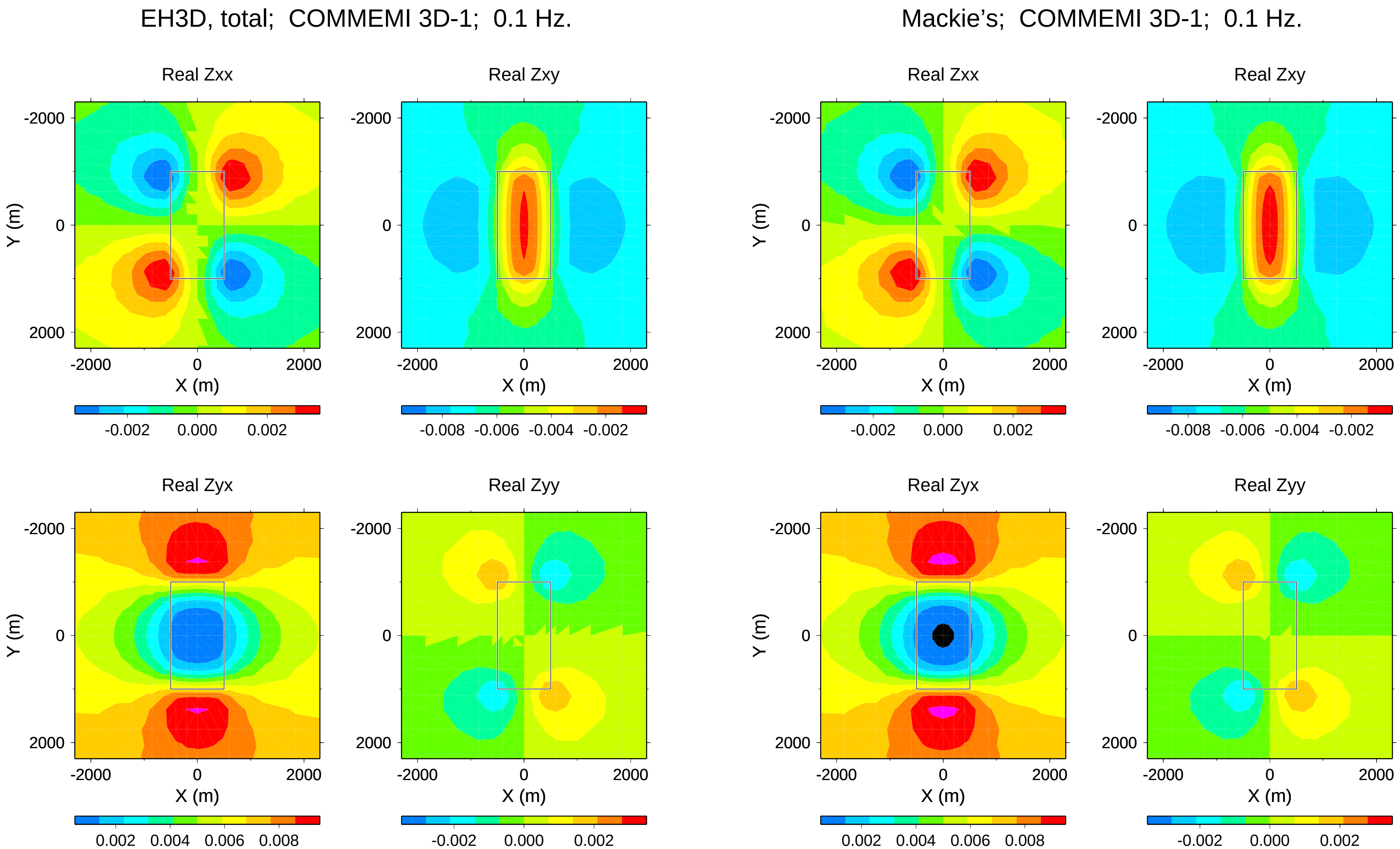
- Both total-field & primary–secondary separation methods have been implemented.

Forward modelling: example

COMMEMI 3D-1 model
(Zhdanov et al., 1997)



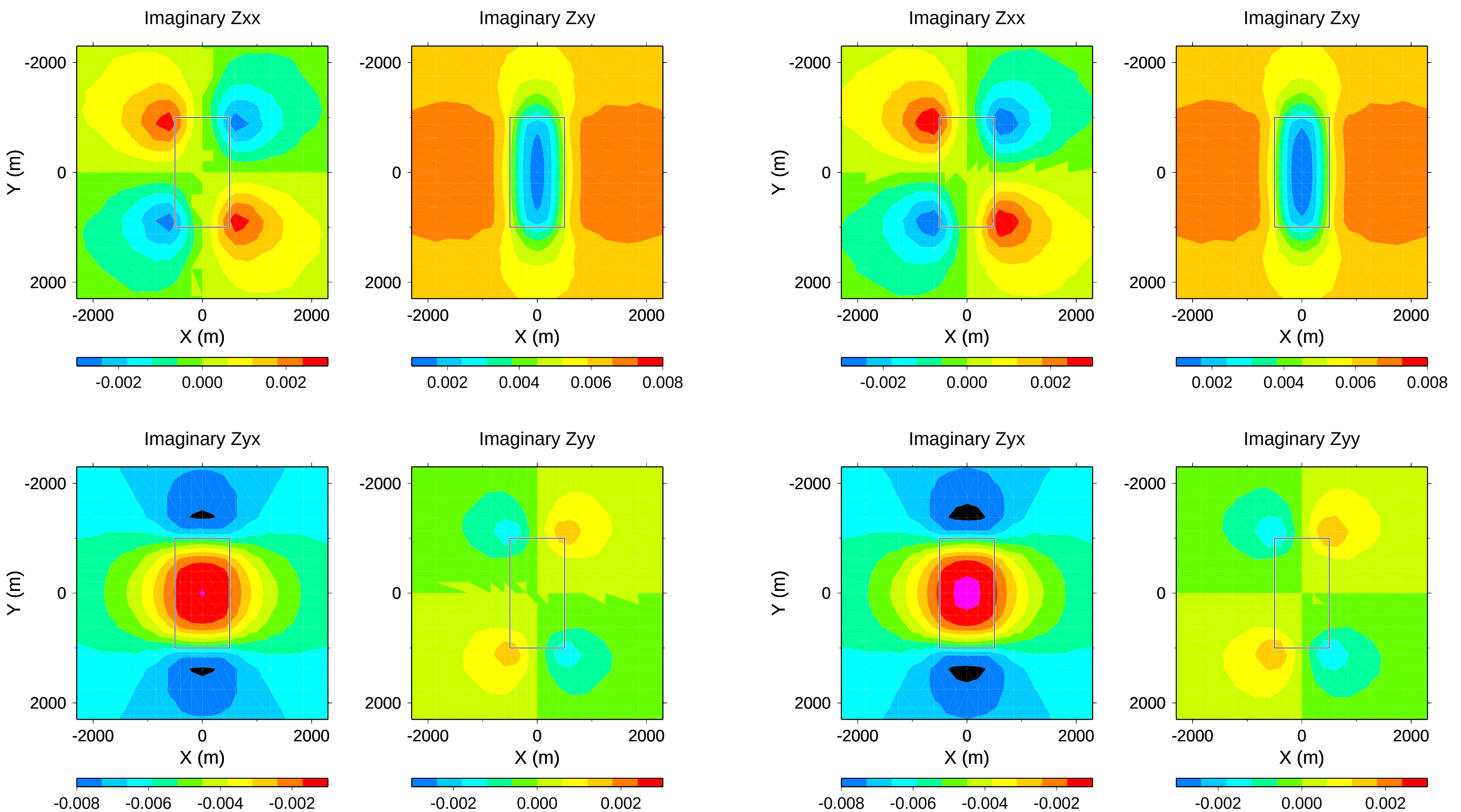
Forward modelling: example



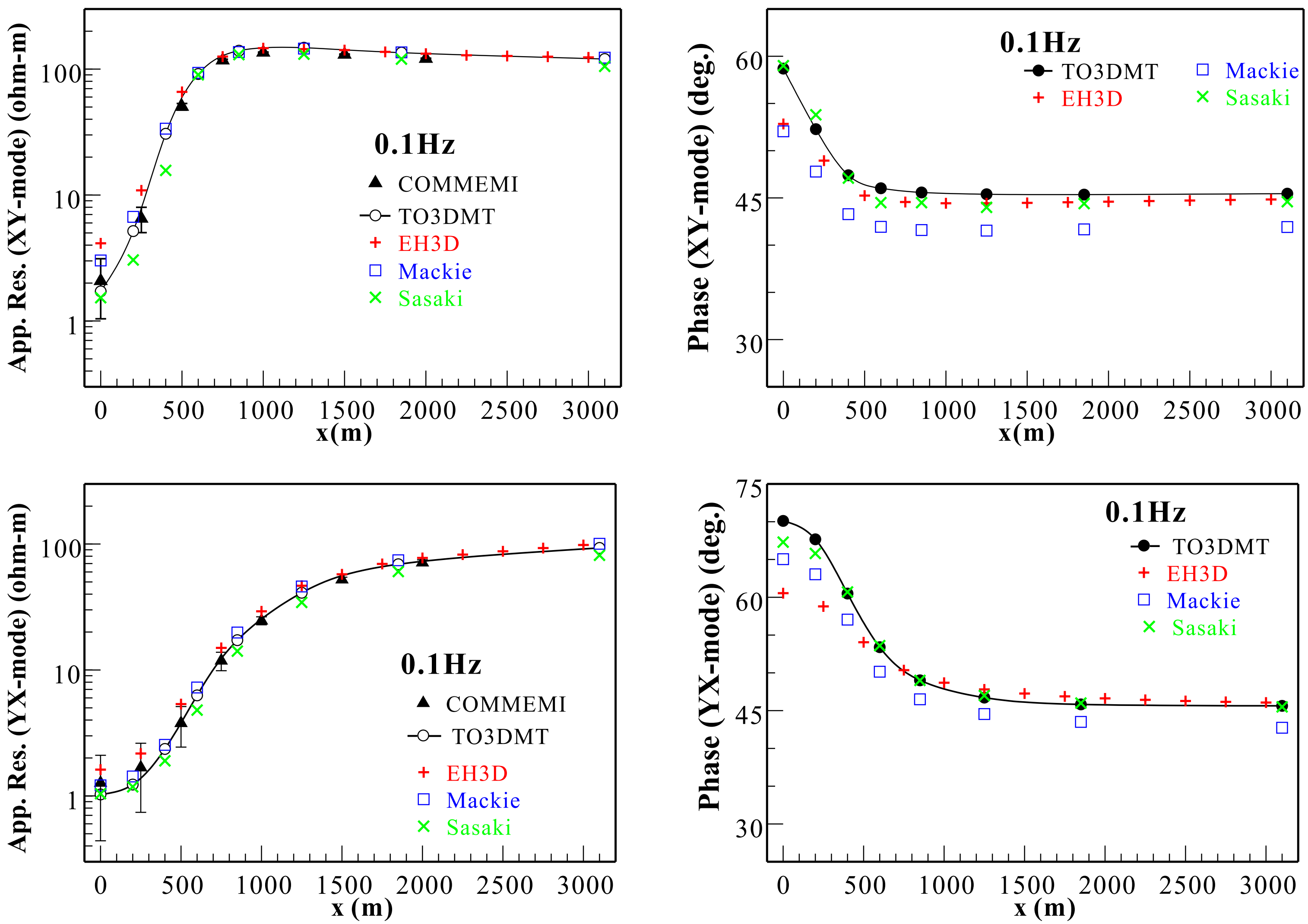
Forward modelling: example

EH3D, total; COMMEMI 3D-1; 0.1 Hz.

Mackie's; COMMEMI 3D-1; 0.1 Hz.



Forward modelling: example



Forward modelling: example

- Mesh: $37 \times 41 \times 24$ cells.
- Memory required: $\sim$250 Mbytes.
- Number of BiCGSTAB iterations: $\sim 100^{\dagger}$.
- Computation time: $\sim$1 minute (per polarisation).

Inversion: equations

- Minimise the objective function:

$$\Phi = \phi_d + \beta\,\phi_m,$$

where ϕ_d & ϕ_m are the typical measures of data-misfit and model structure, and β is the trade-off parameter.

- Solve with iterative, Gauss-Newton minimisation procedure:

$$\left(\mathbf{J}^T\mathbf{W}_d^T\mathbf{W}_d\mathbf{J} + \beta\mathbf{W}^T\mathbf{W}\right)\delta\mathbf{m} = \\ -\mathbf{J}^T\mathbf{W}_d^T\mathbf{W}_d\left(\mathbf{d}^{\mathrm{obs}} - \mathbf{d}^n\right) - \beta\mathbf{W}^T\mathbf{W}\left(\mathbf{m}^n - \mathbf{m}^{\mathrm{ref}}\right).$$

Inversion: equations

- Data are impedances, or apparent resistivities & phases, and can be represented as

$$d_i \;=\; \mathcal{F}_i\left[\, \mathbf{Q}\big(\mathbf{u}_p^{(1)} + \mathbf{u}_s^{(1)}\big),\, \mathbf{Q}\big(\mathbf{u}_p^{(2)} + \mathbf{u}_s^{(2)}\big)\,\right],$$

where $\mathbf{Q}$ is the interpolation matrix, and $\mathcal{F}_i$ represents the calculation of impedances from E & H.

- The Jacobian matrix of sensitivities is represented by

$$\mathbf{J} \;=\; \mathbf{S}^{(1)}\mathbf{Q}\mathcal{A}^{-1}\Big(\mathbf{G}_p^{(1)} - \mathbf{G}_s^{(1)}\Big) \;+\; \mathbf{S}^{(2)}\mathbf{Q}\mathcal{A}^{-1}\Big(\mathbf{G}_p^{(2)} - \mathbf{G}_s^{(2)}\Big).$$

Inversion: algorithm

- System of equations solved using an inexact preconditioned conjugate algorithm with ILU decomposition of $(\mathbf{W}^T\mathbf{W} + 0.1\mathbf{I})$ as the preconditioner (Haber, Ascher & Oldenburg, 2002).

- This requires only the product of $\mathbf{J}$ or $\mathbf{J}^T$ with a vector (Haber, Ascher & Oldenburg, 2000).

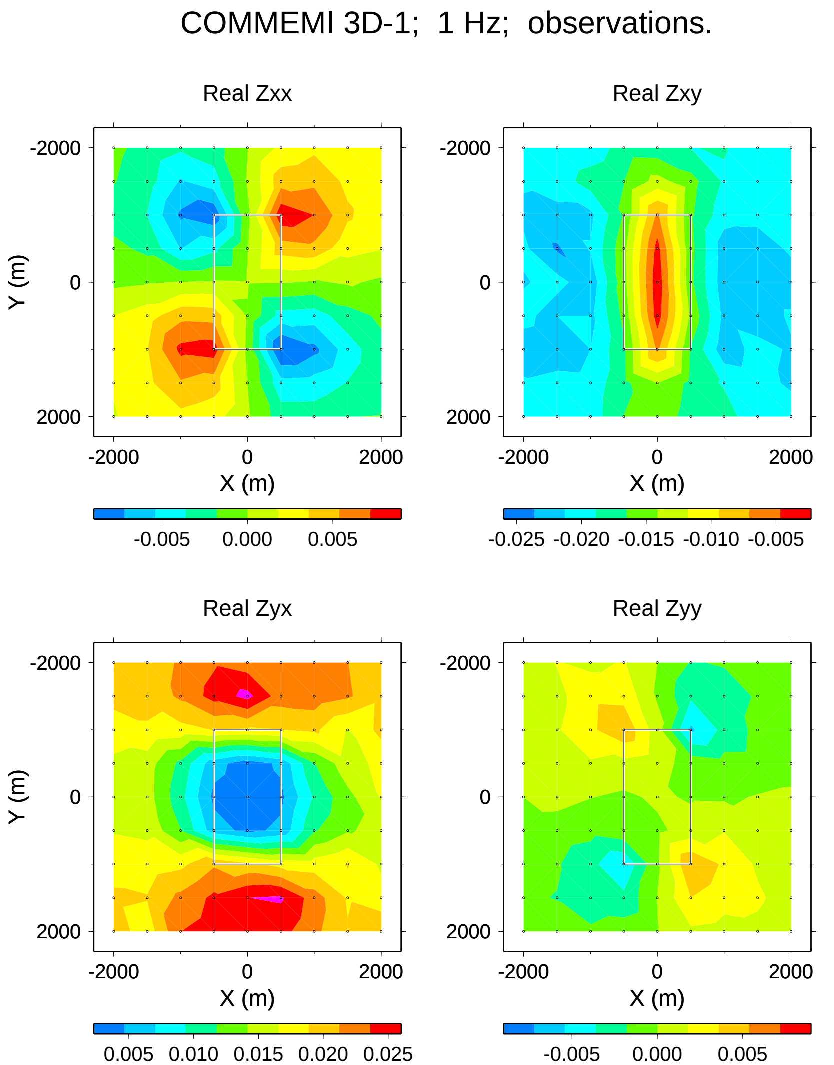
- These operations can be done efficiently using the forward modelling algorithm with a modified right-hand side (Mackie & Madden, 1993).

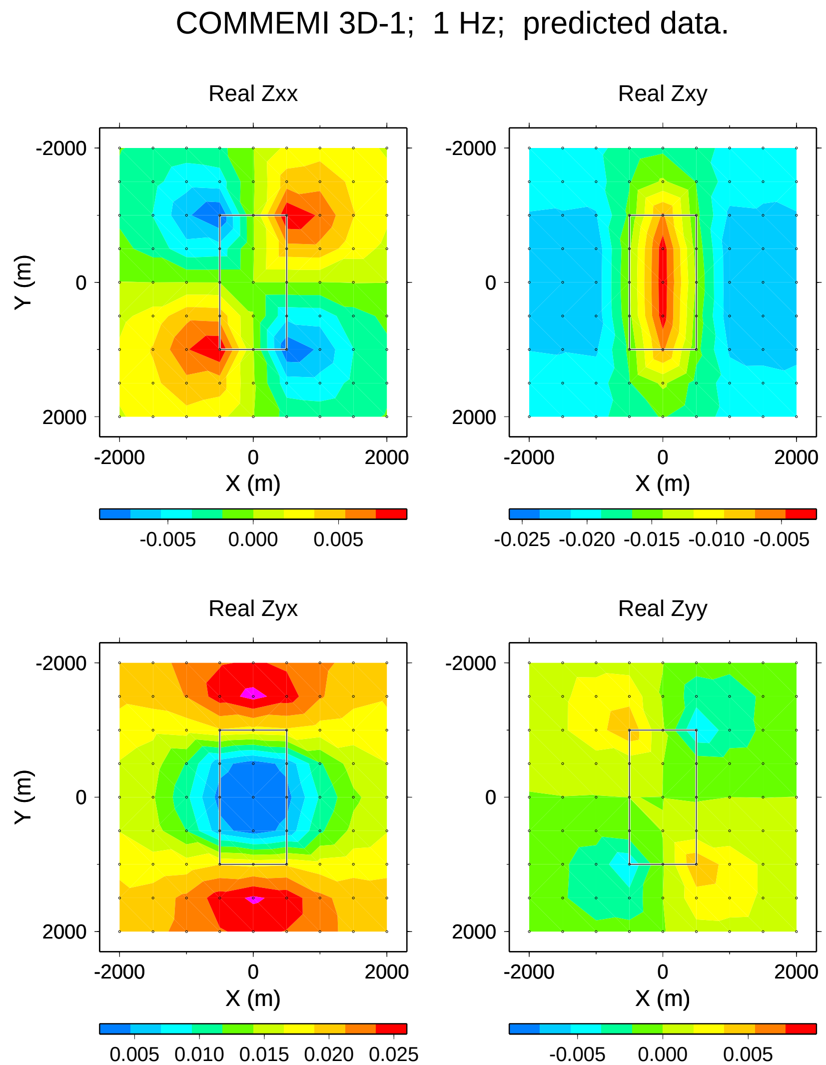
- Prescribed "cooling" schedule for the trade-off parameter.

Inversion: example

- Synthetic data generated from the COMMEMI 3D-1 model:
  - 81 observation locations on a regular grid;
  - 5 frequencies (0.1, 0.316, 1.0, 3.16 & 10 Hz);
  - real & imaginary parts of all four components of the impedance tensor;
  - a total of 3240 data;
  - noise with standard deviation equal to 5% of a datum, plus a threshold, was added.

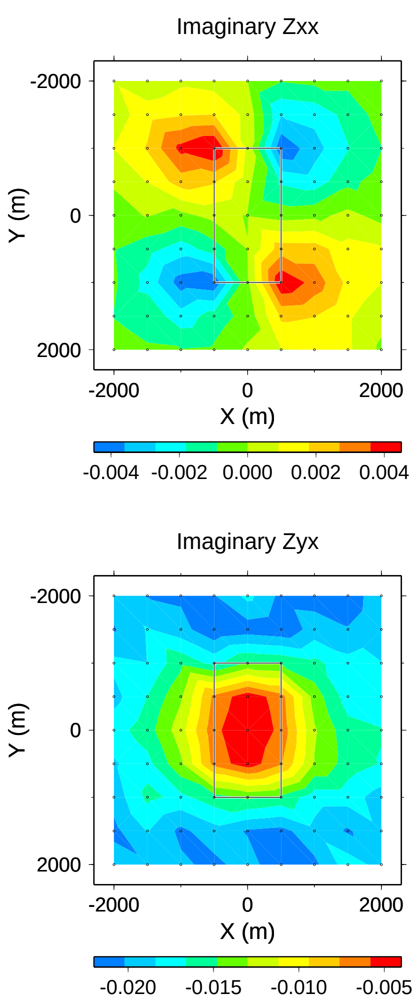
Inversion example: observed & predicted data



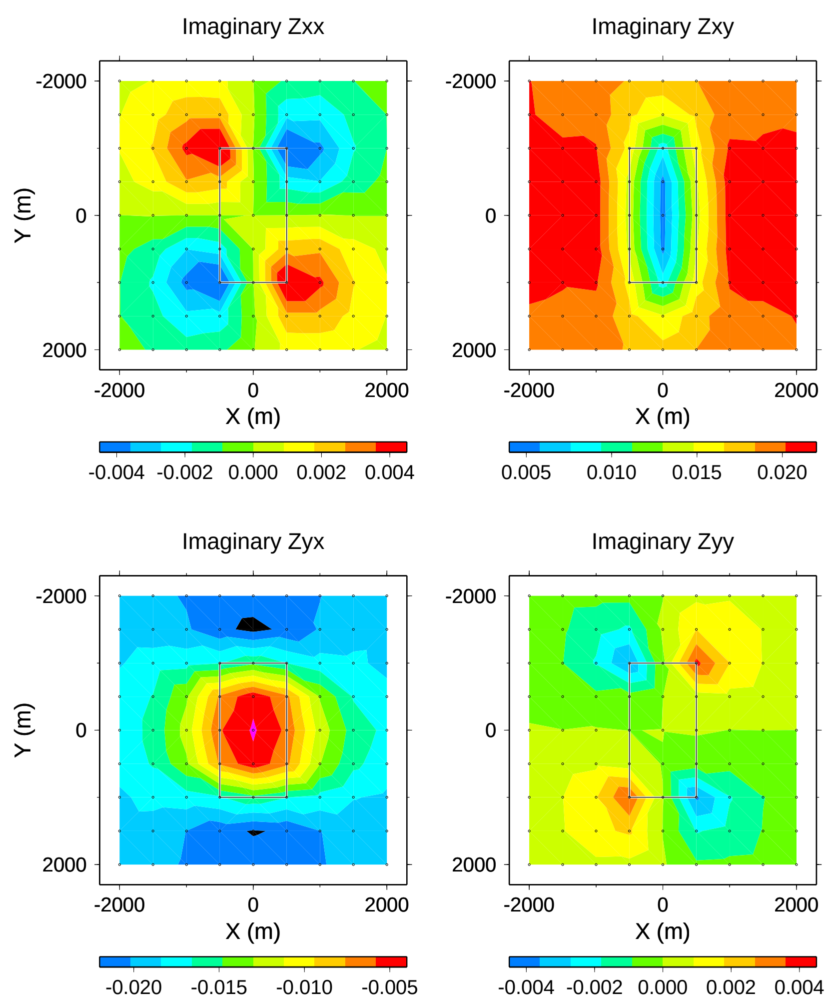


Inversion example: observed & predicted data

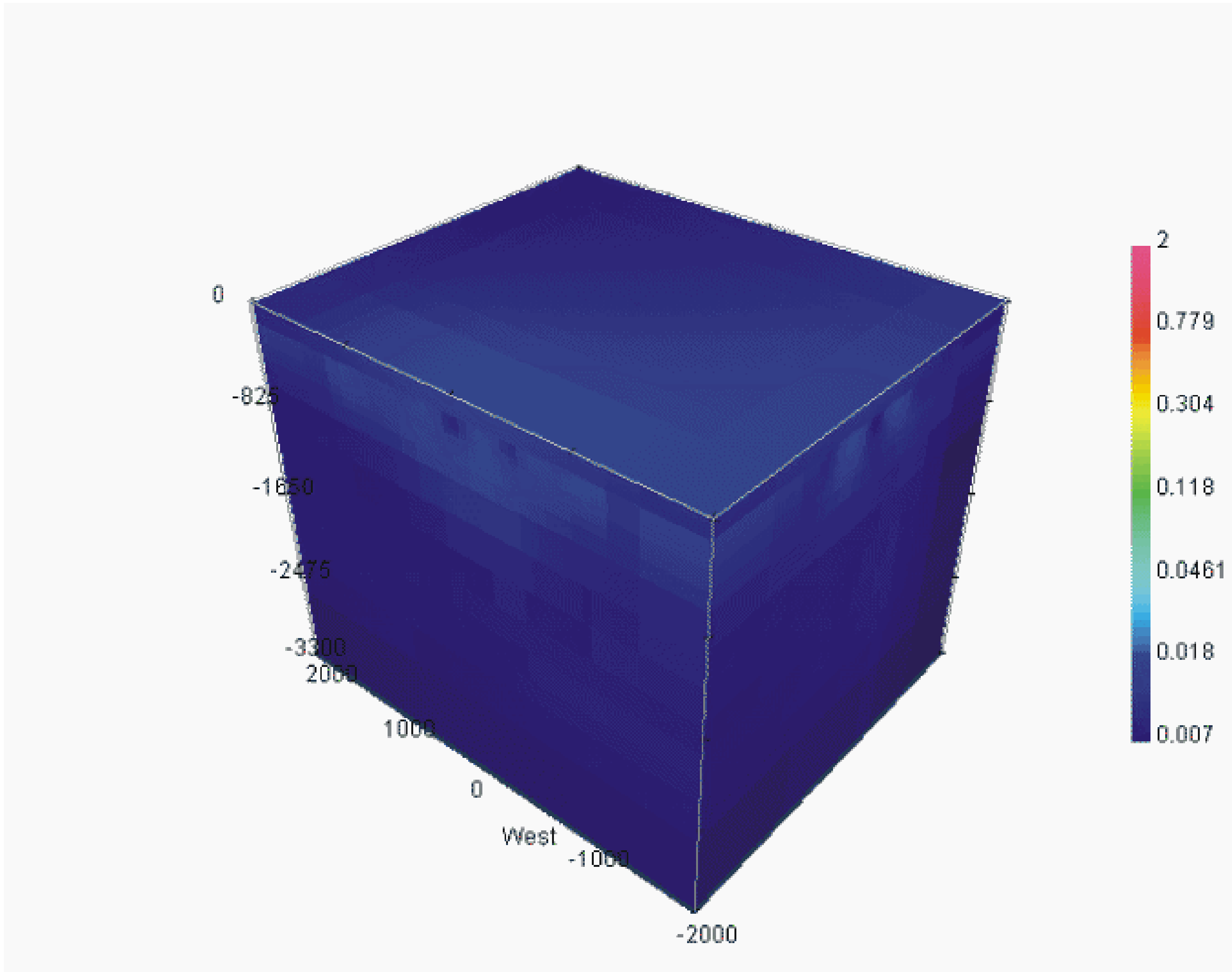
COMMEMI 3D-1; 1 Hz; observations.



COMMEMI 3D-1; 1 Hz; predicted data.



Inversion example: constructed model



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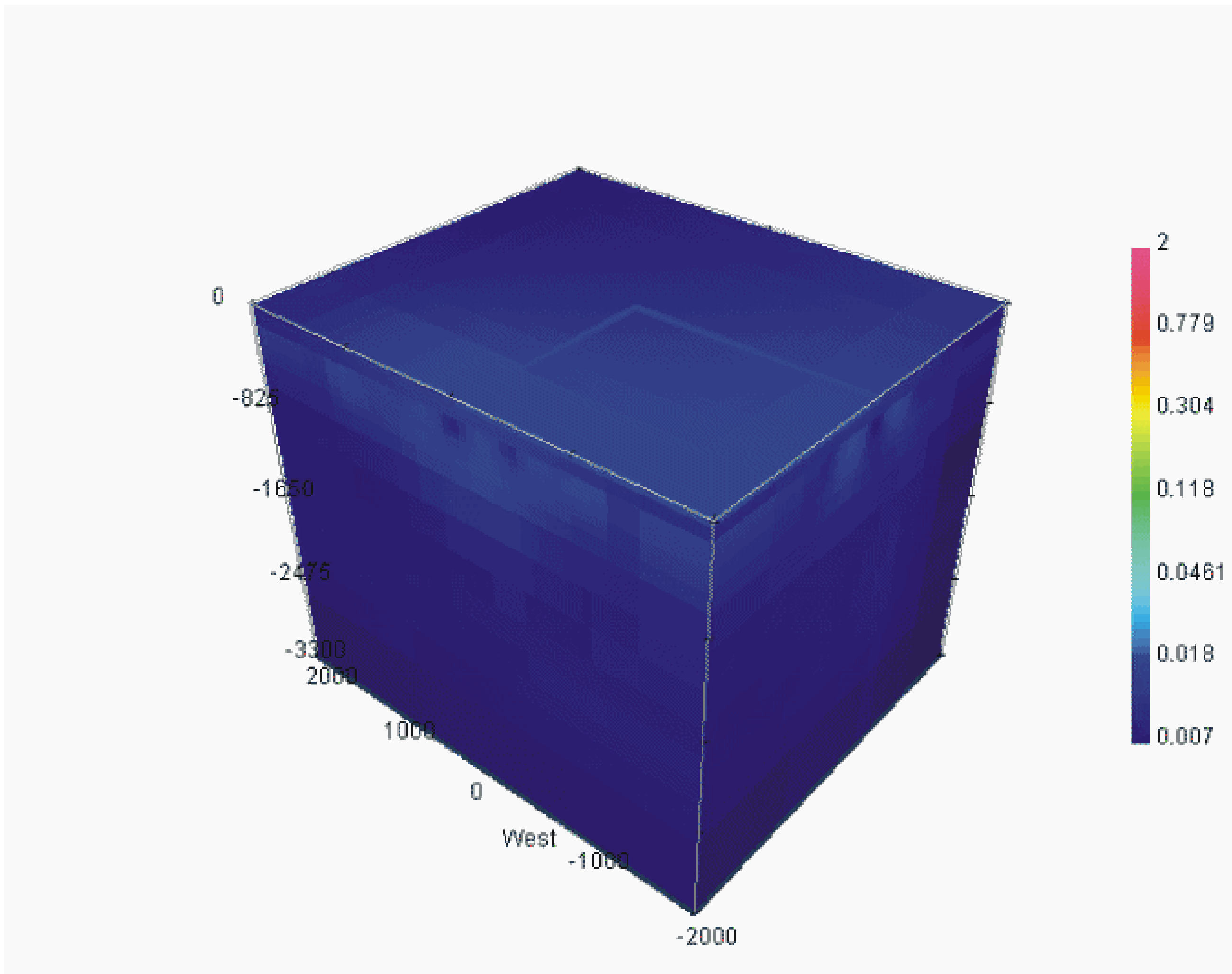

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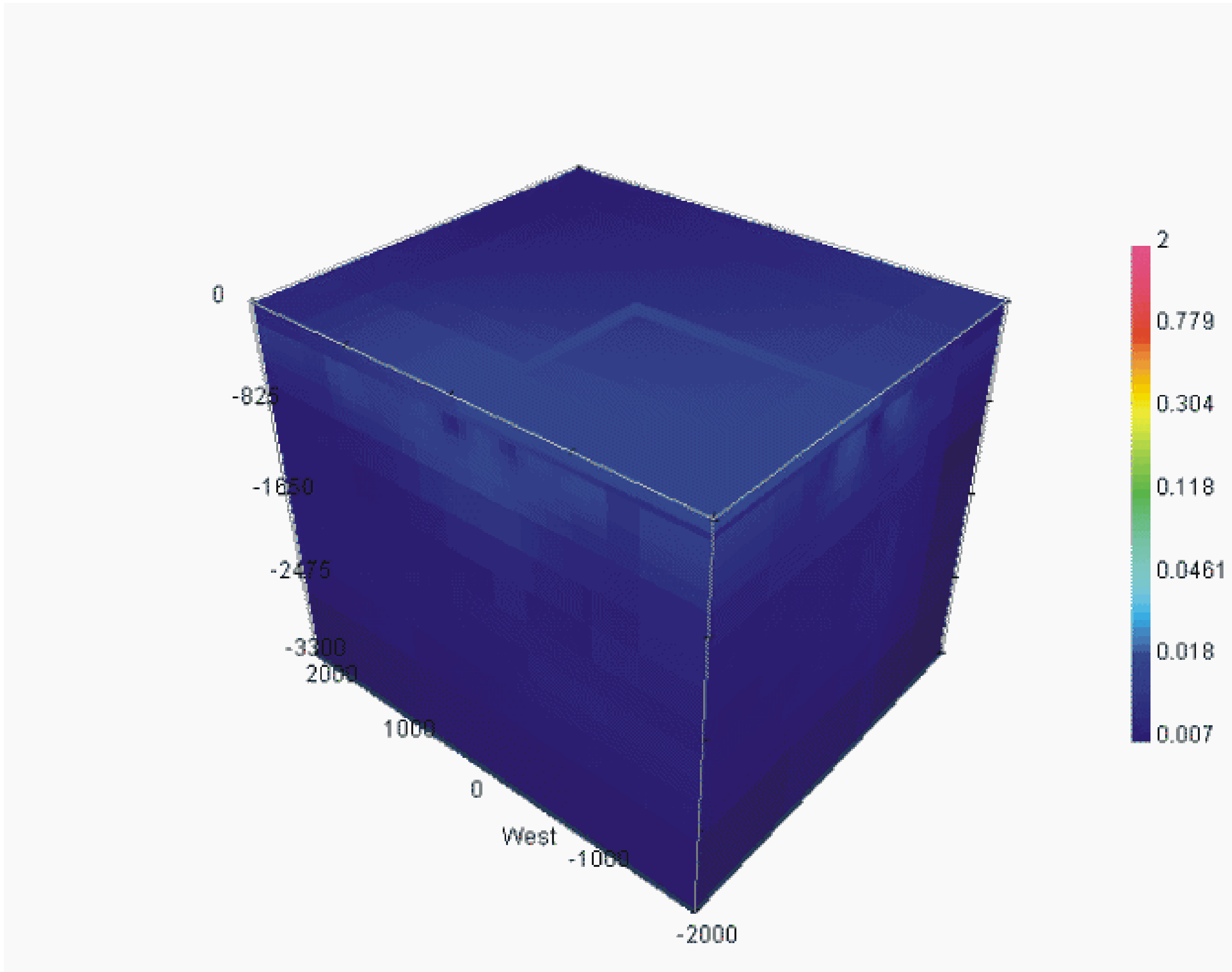

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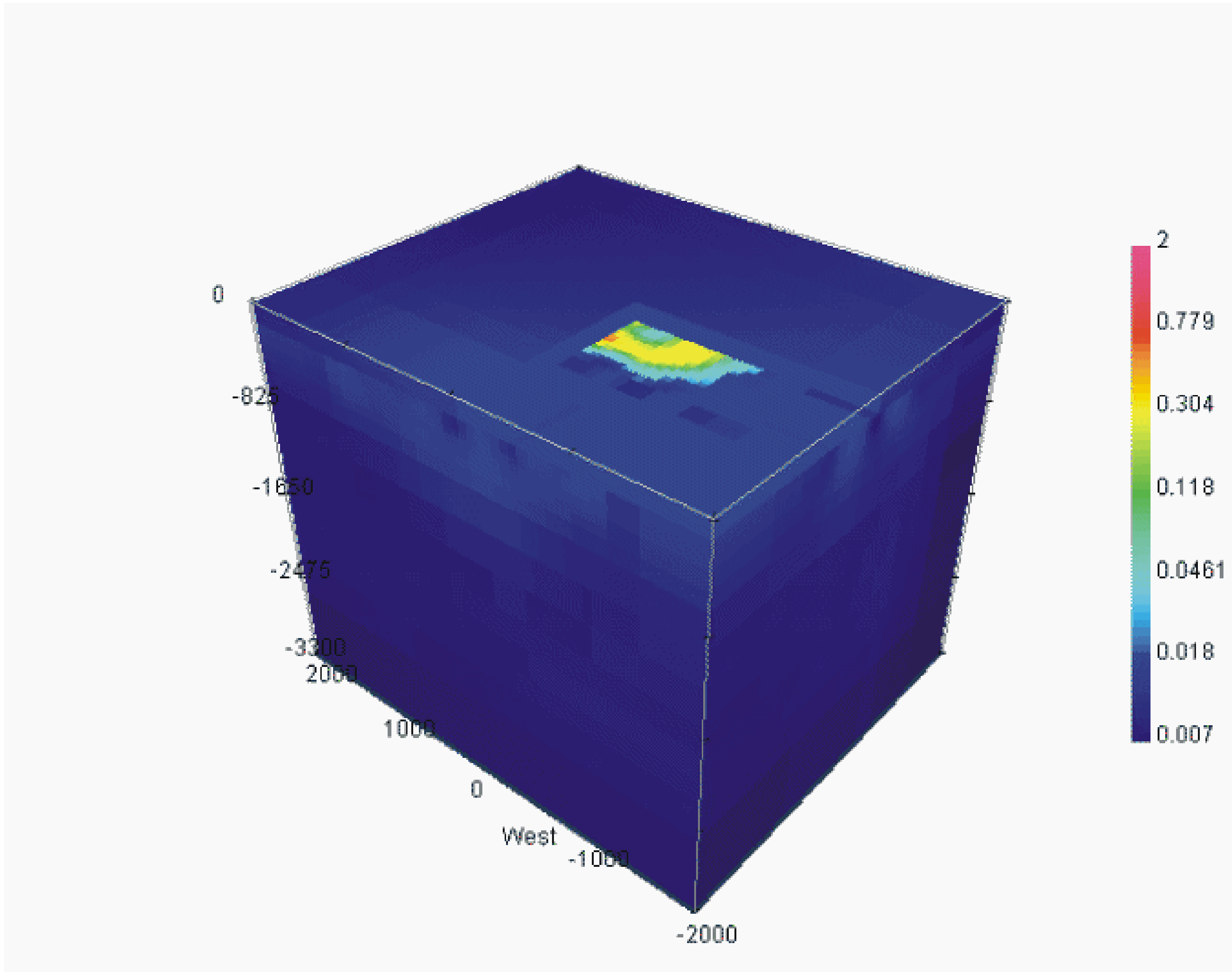

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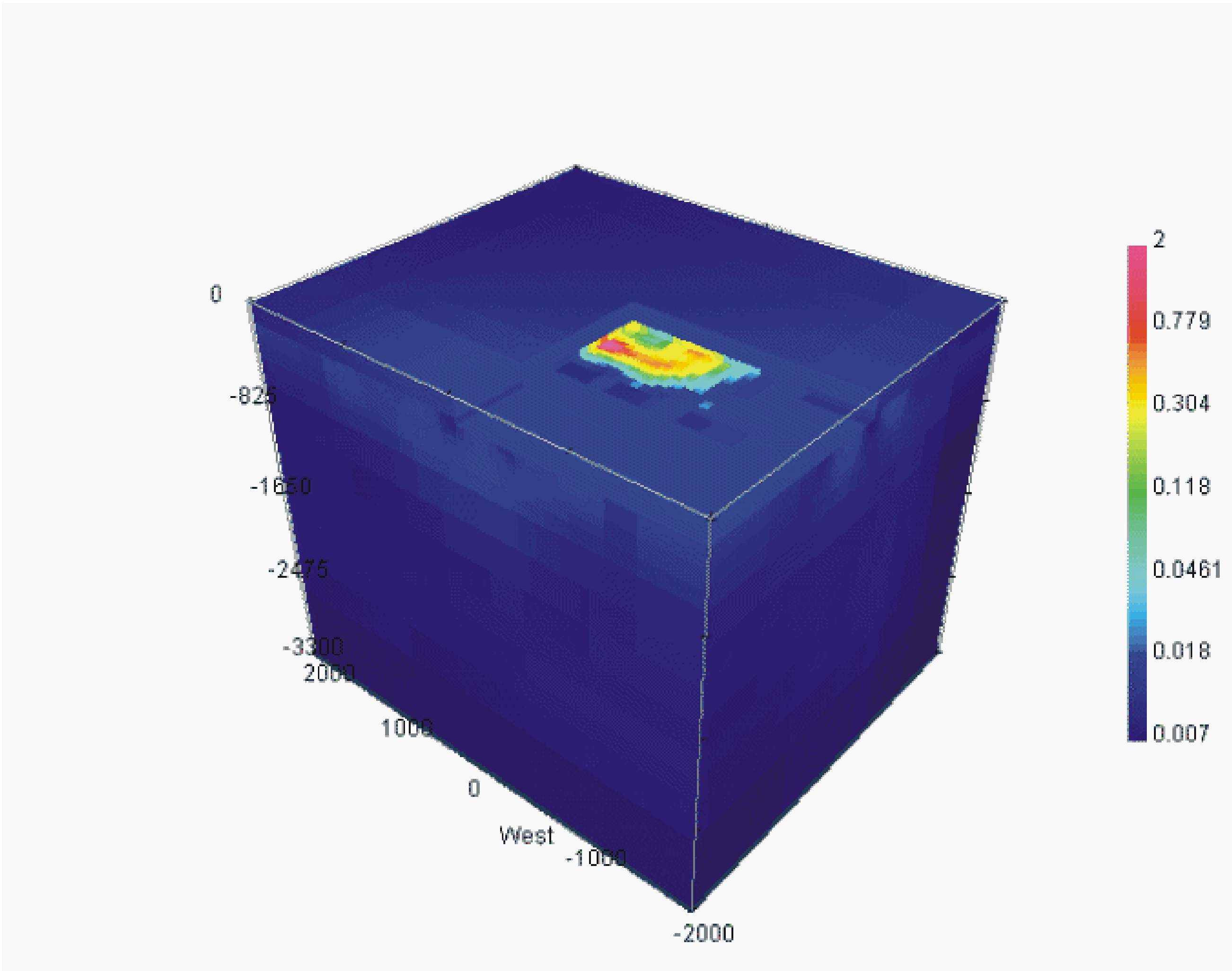

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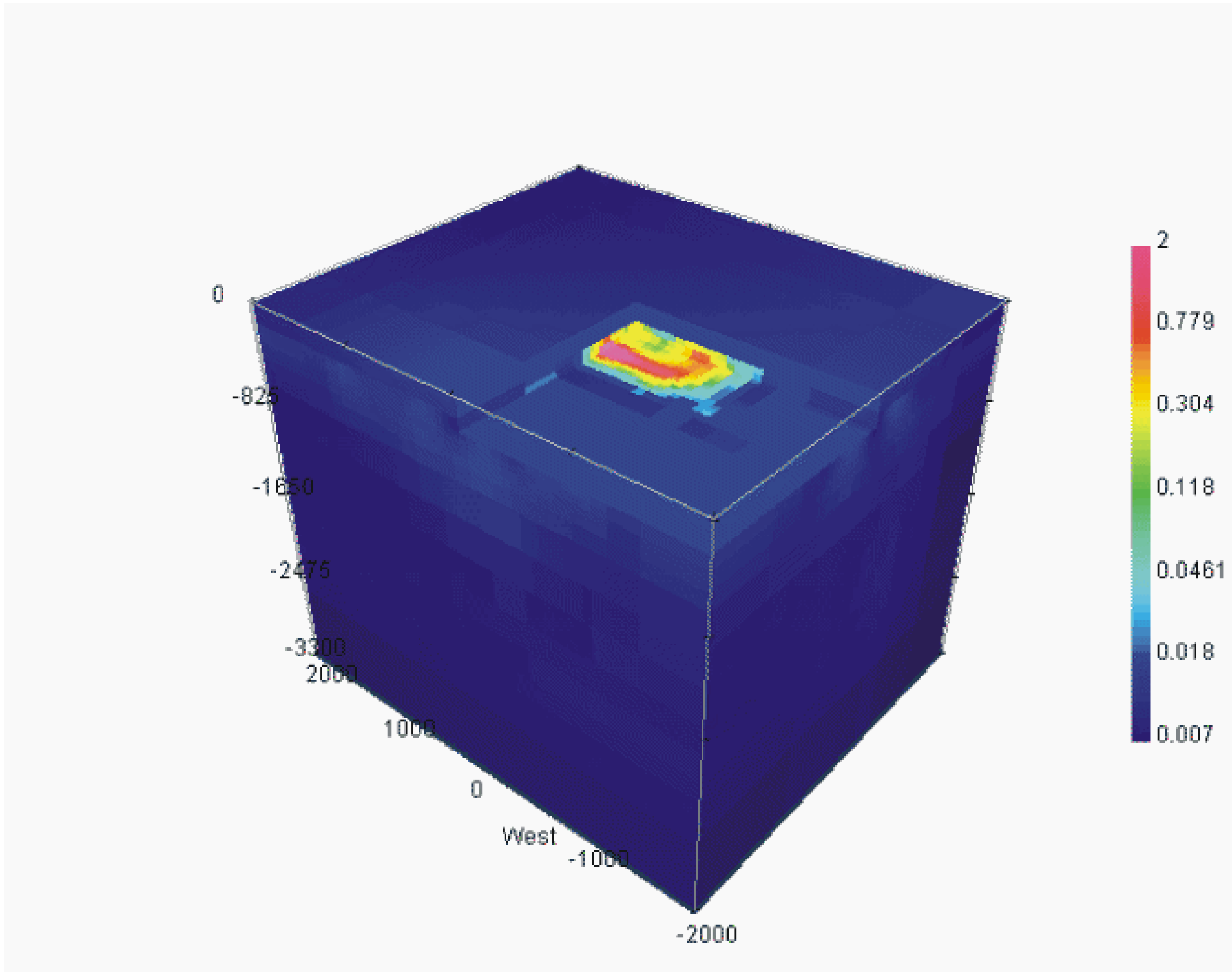

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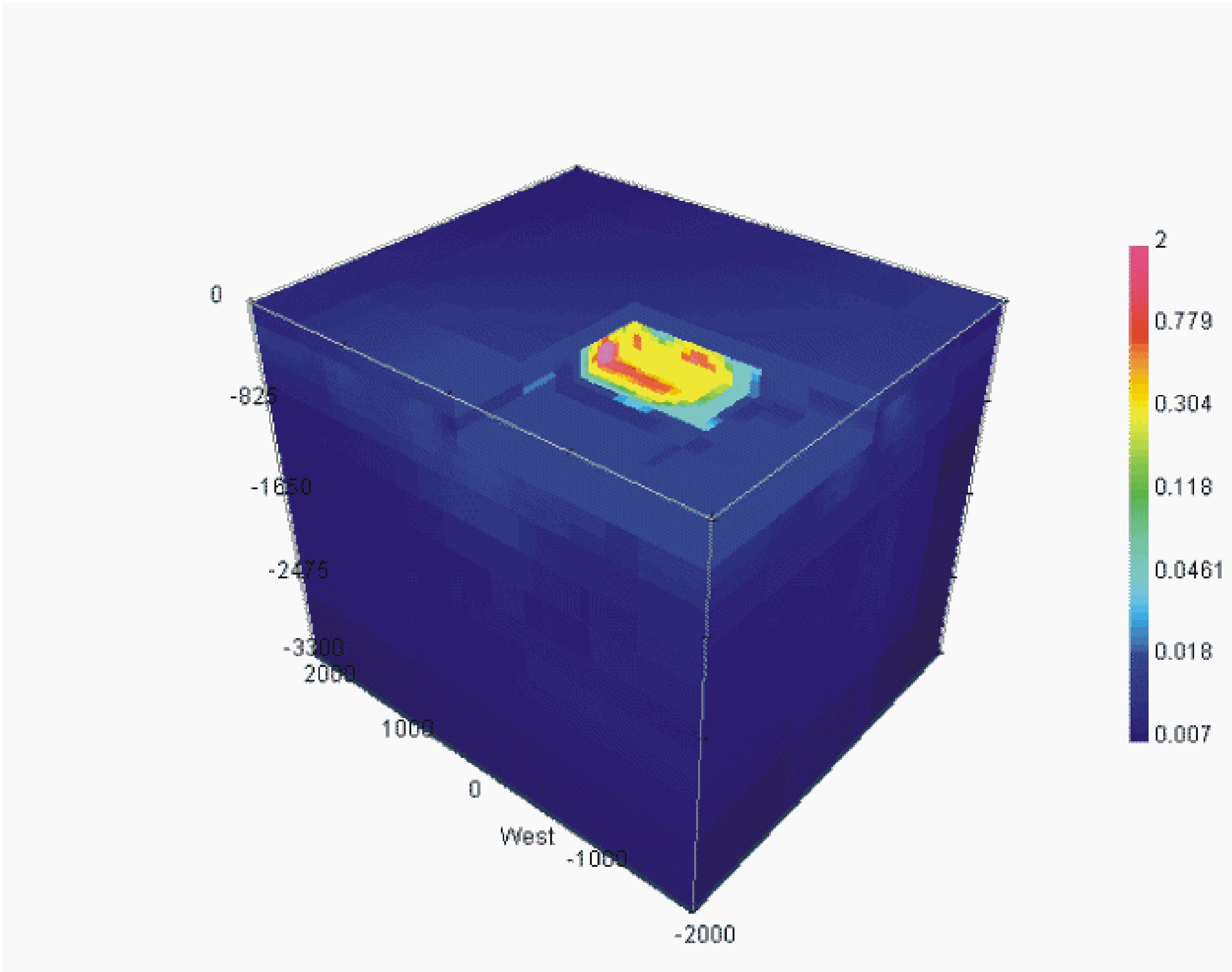

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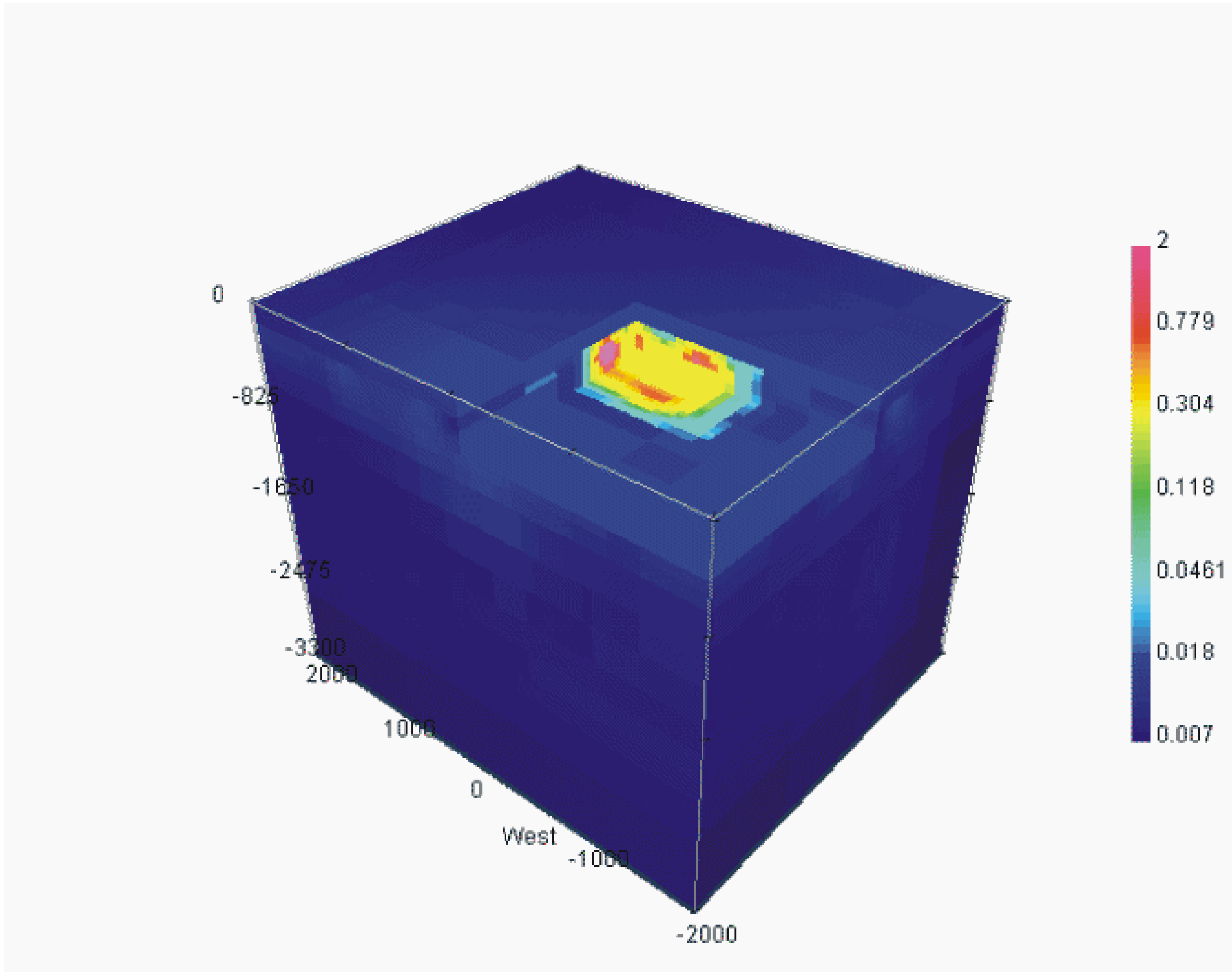

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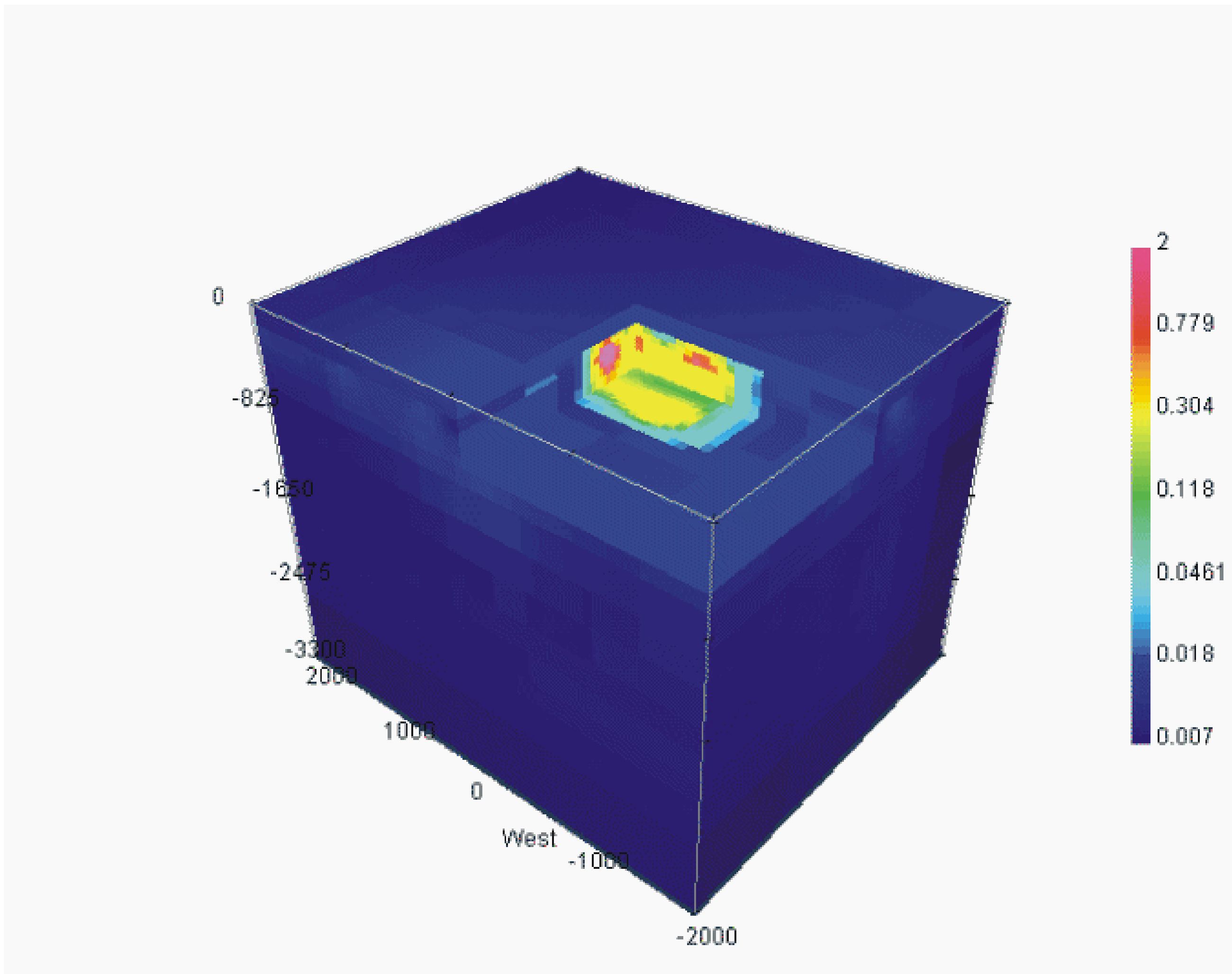

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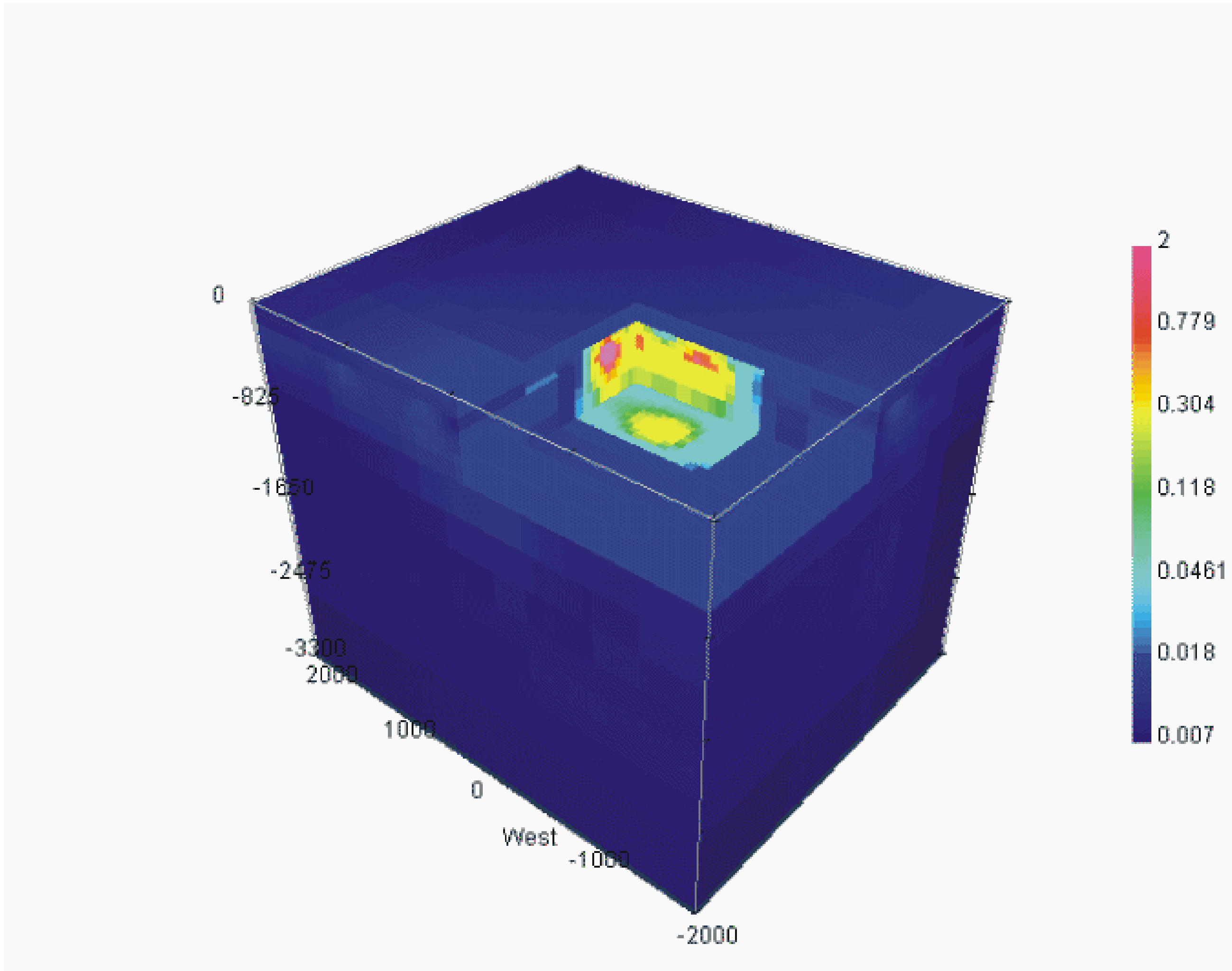

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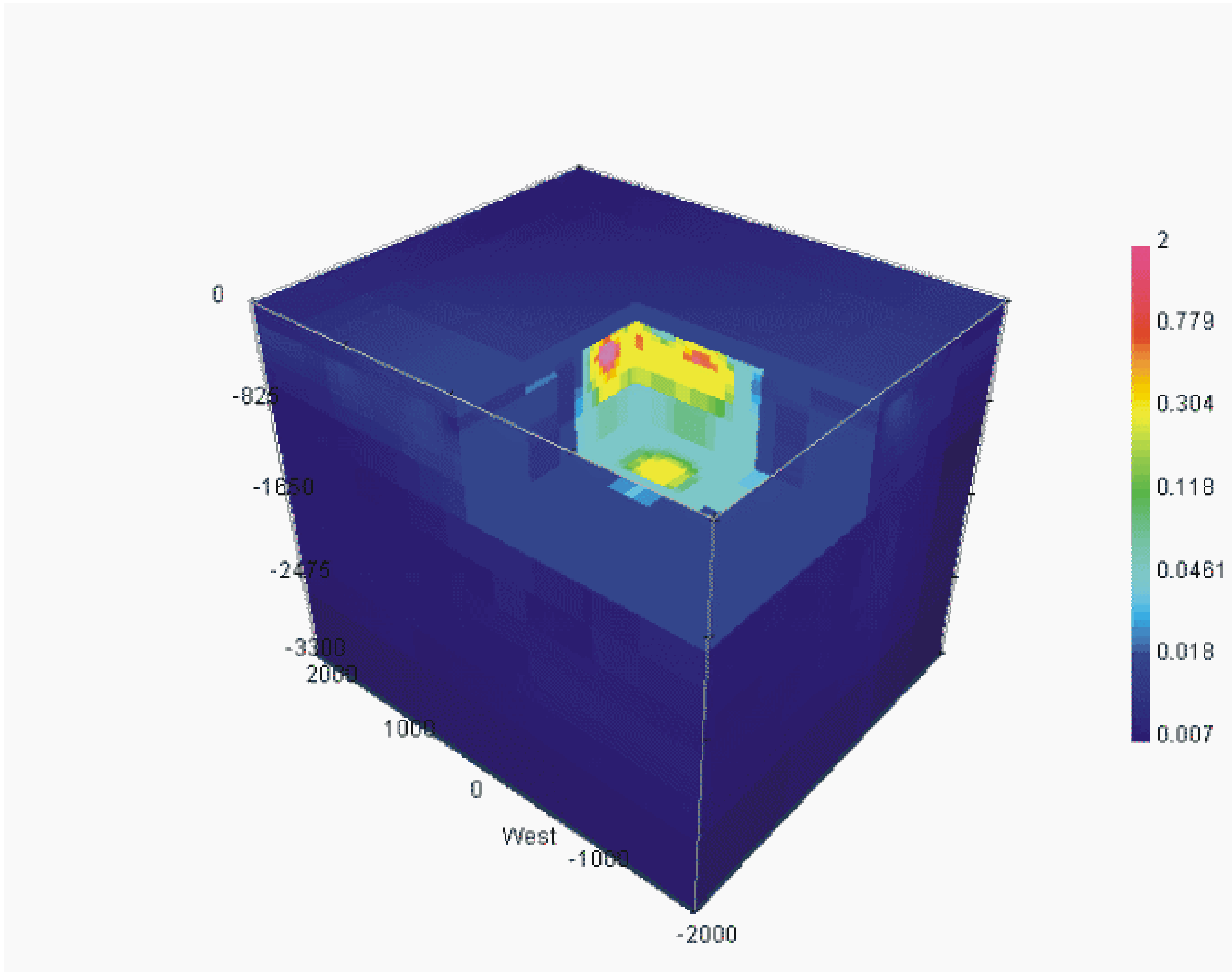

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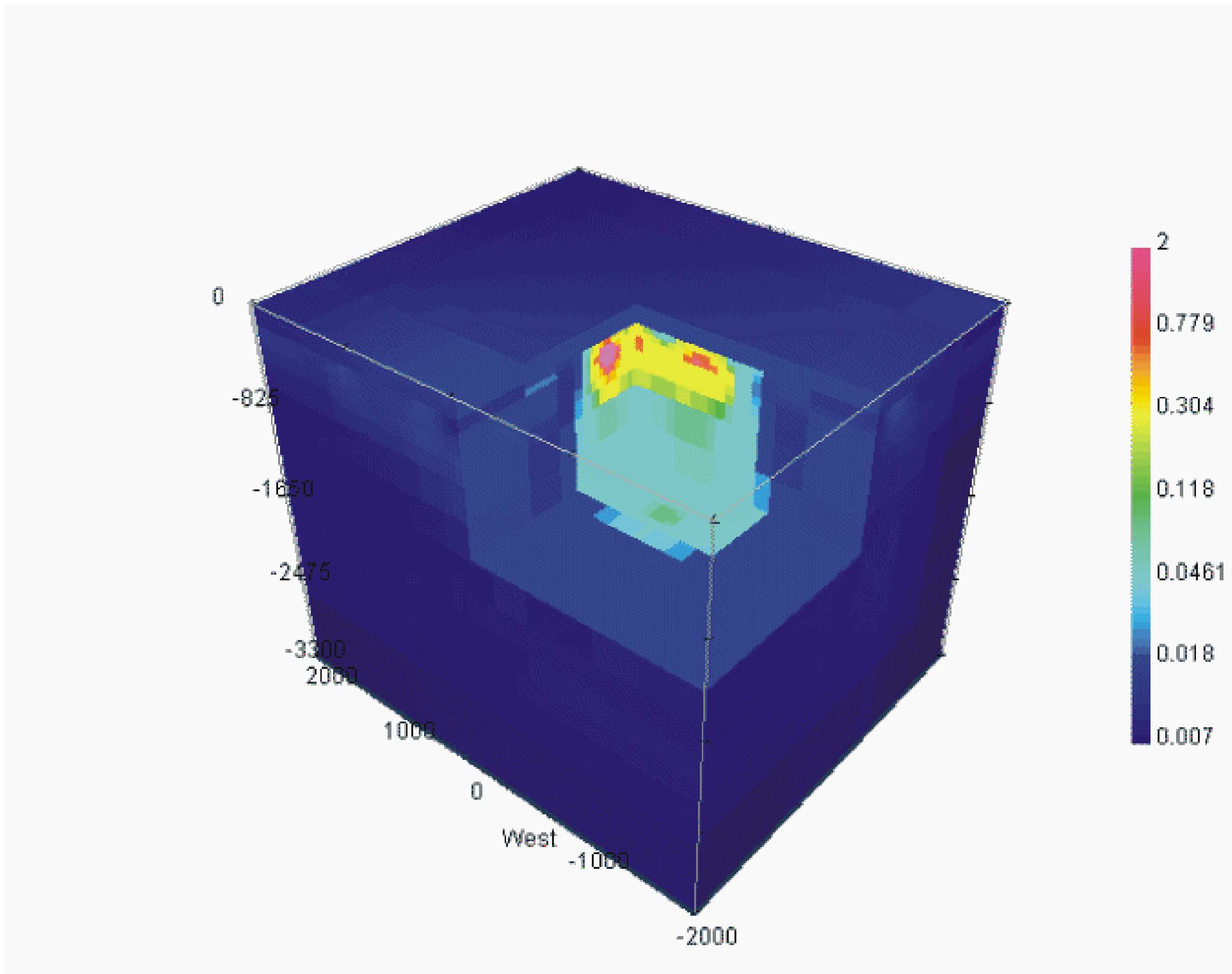

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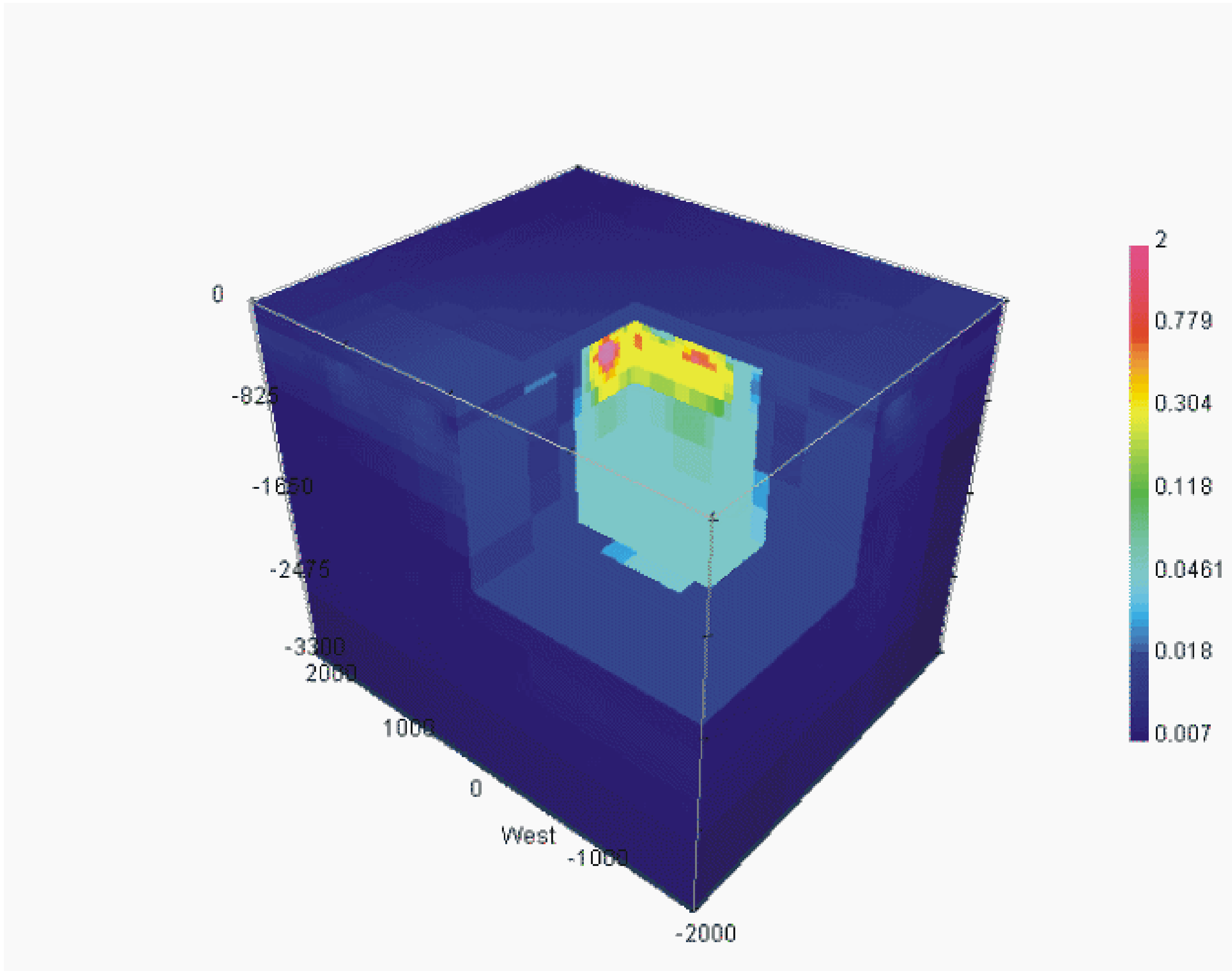

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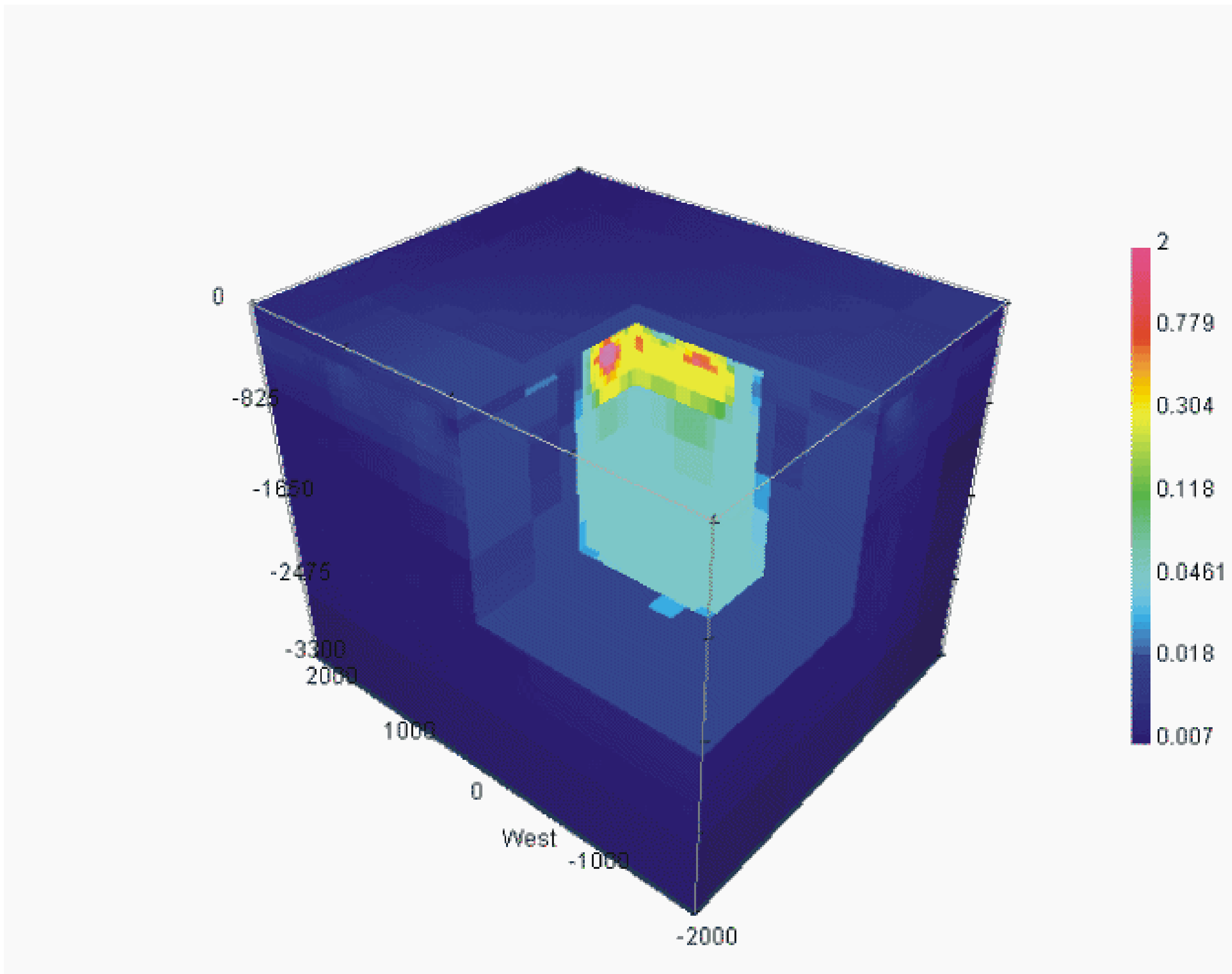

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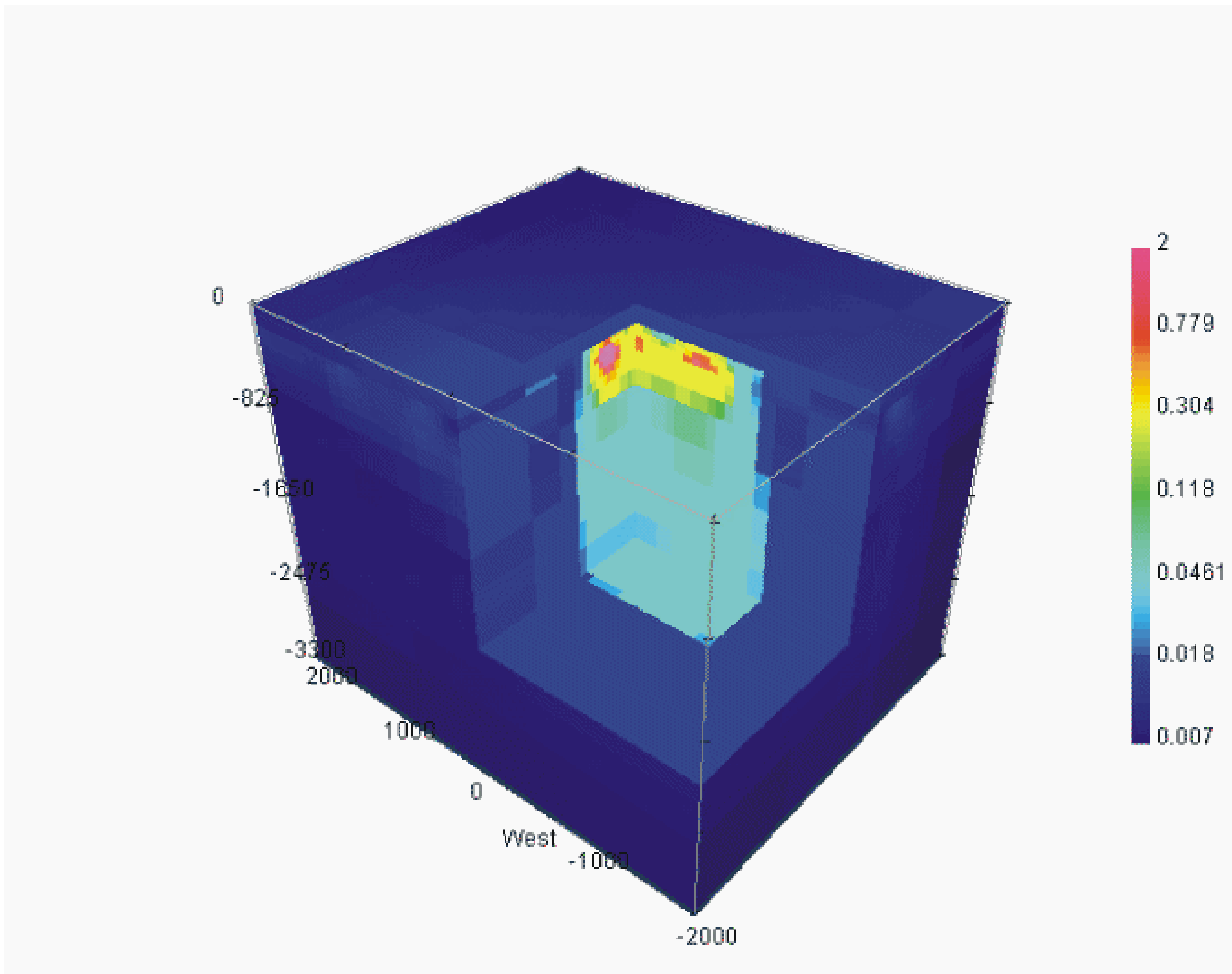

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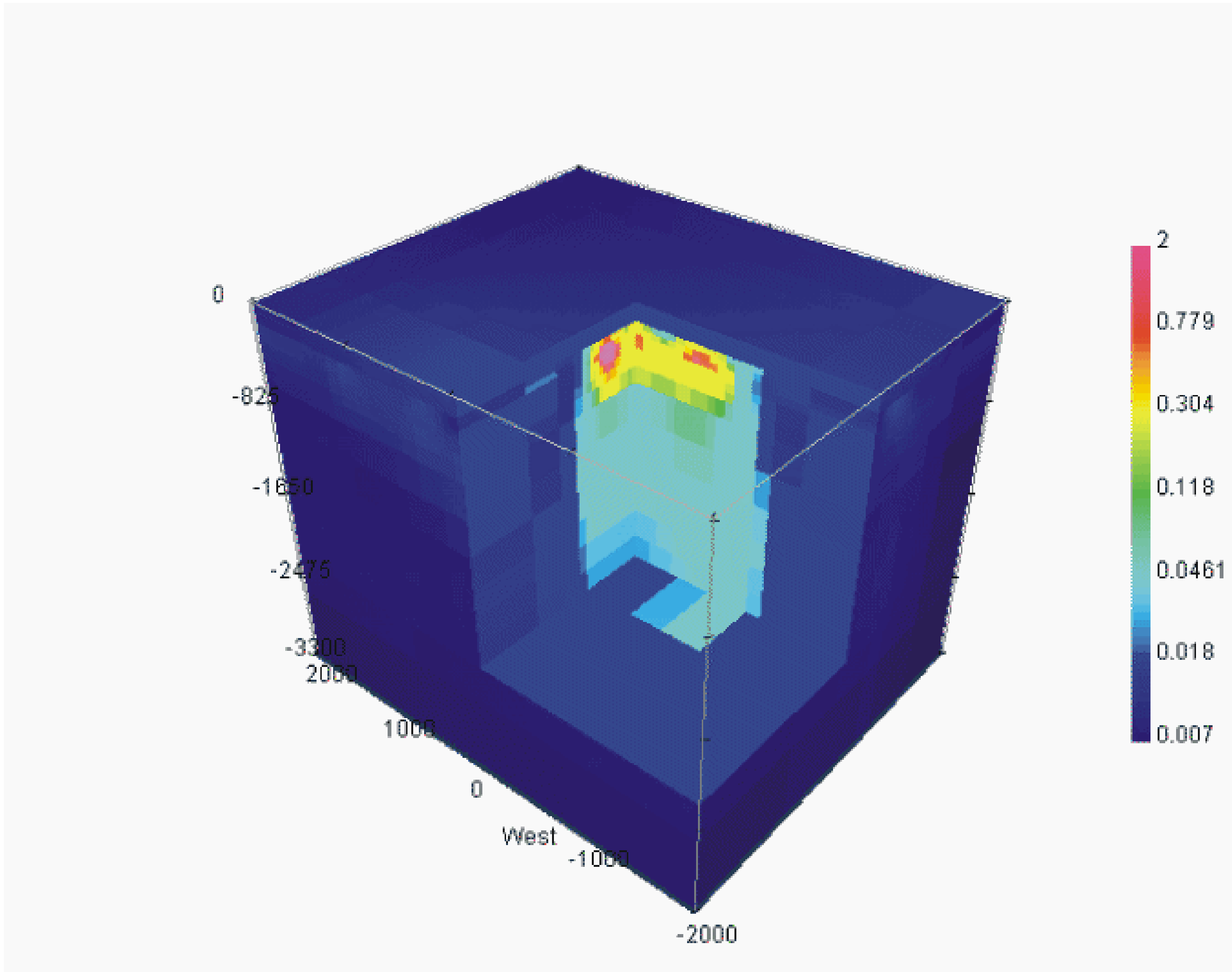

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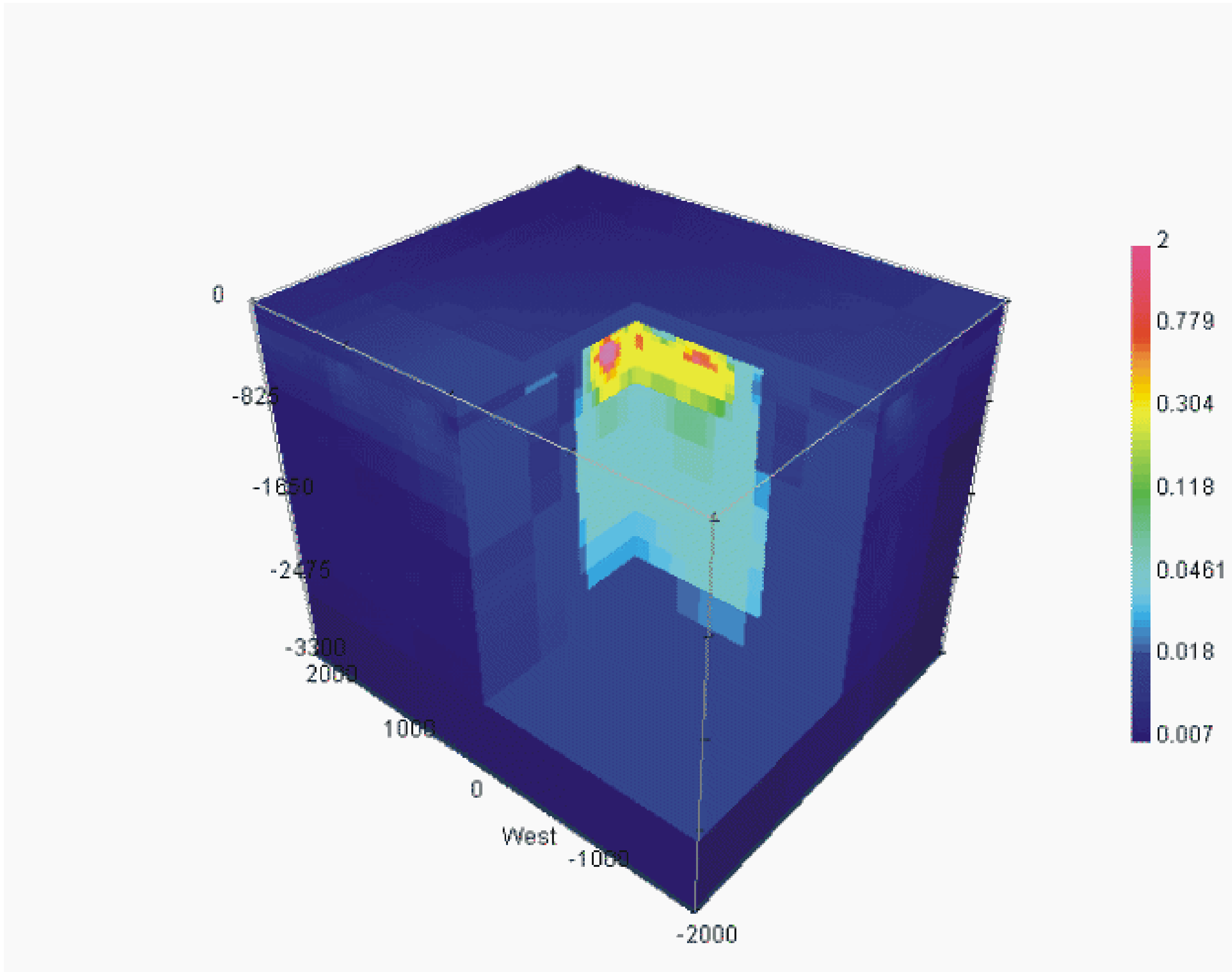

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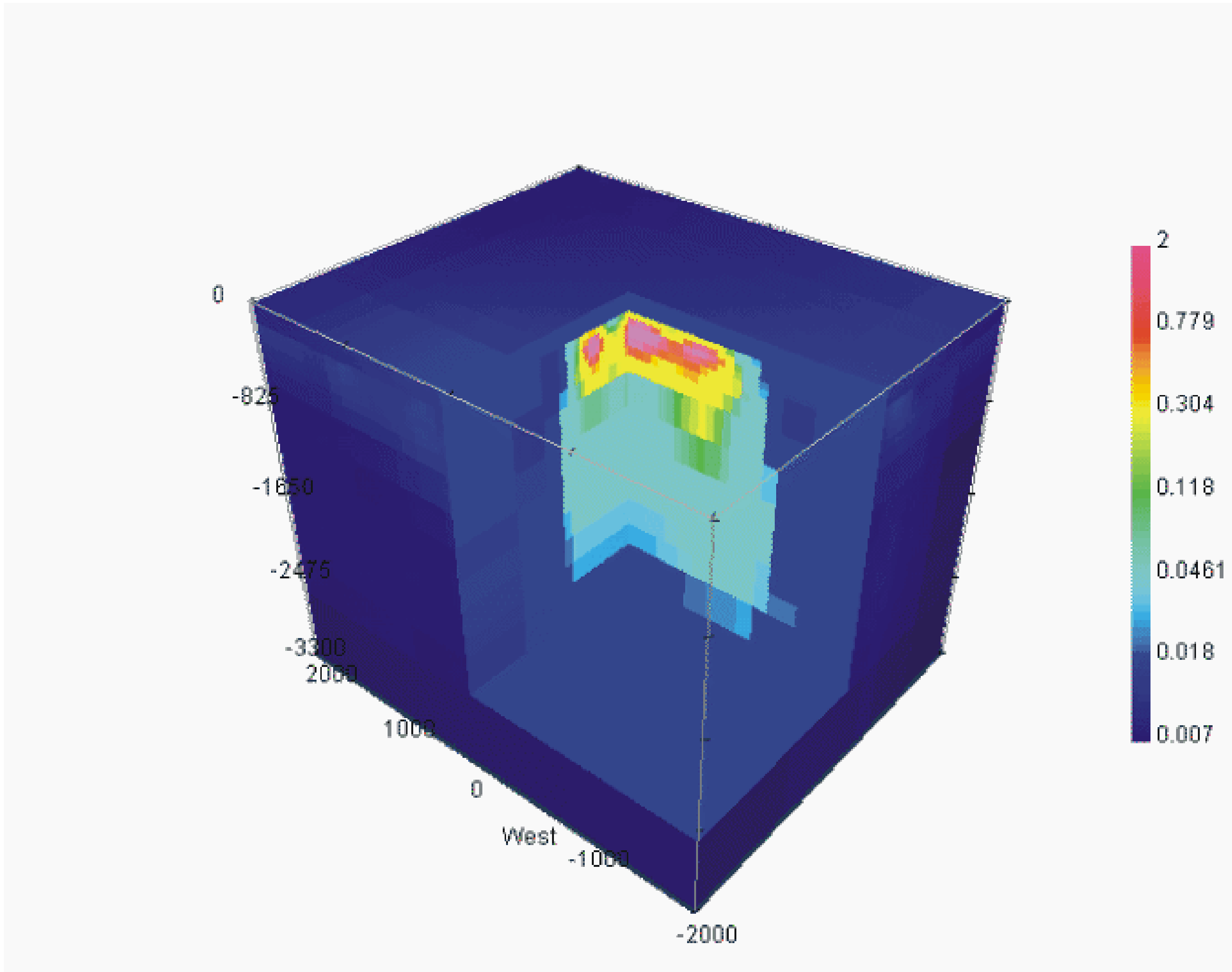

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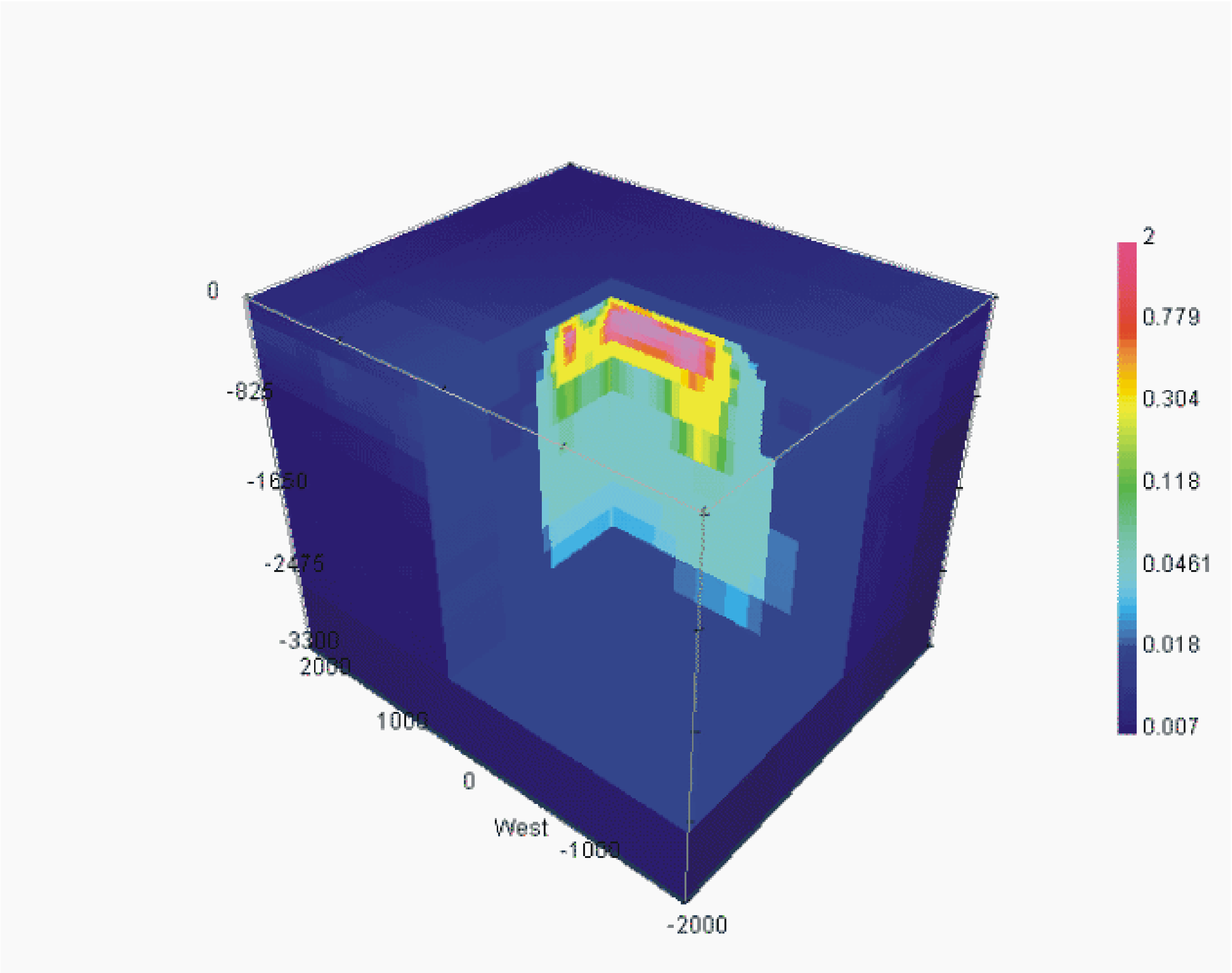

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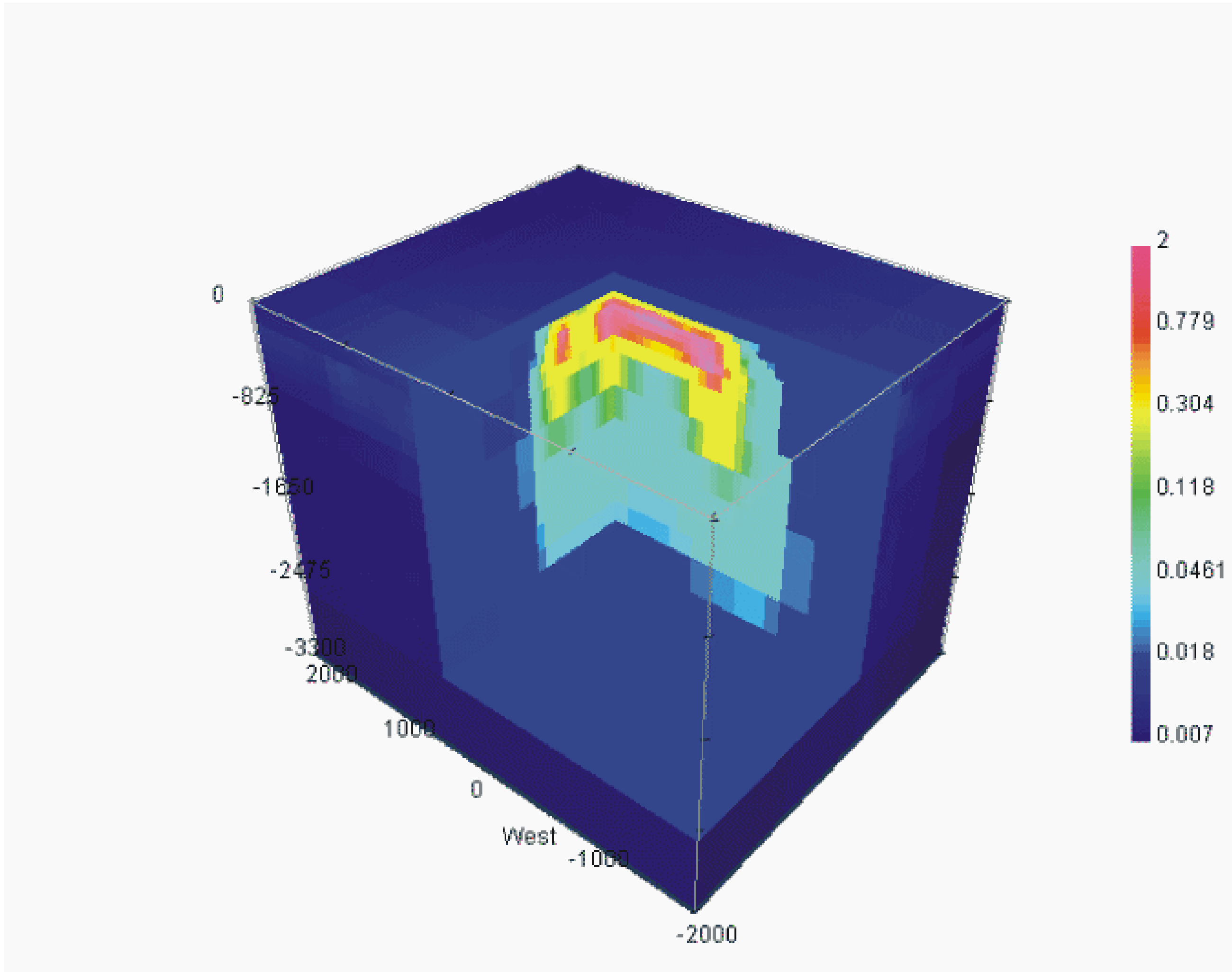
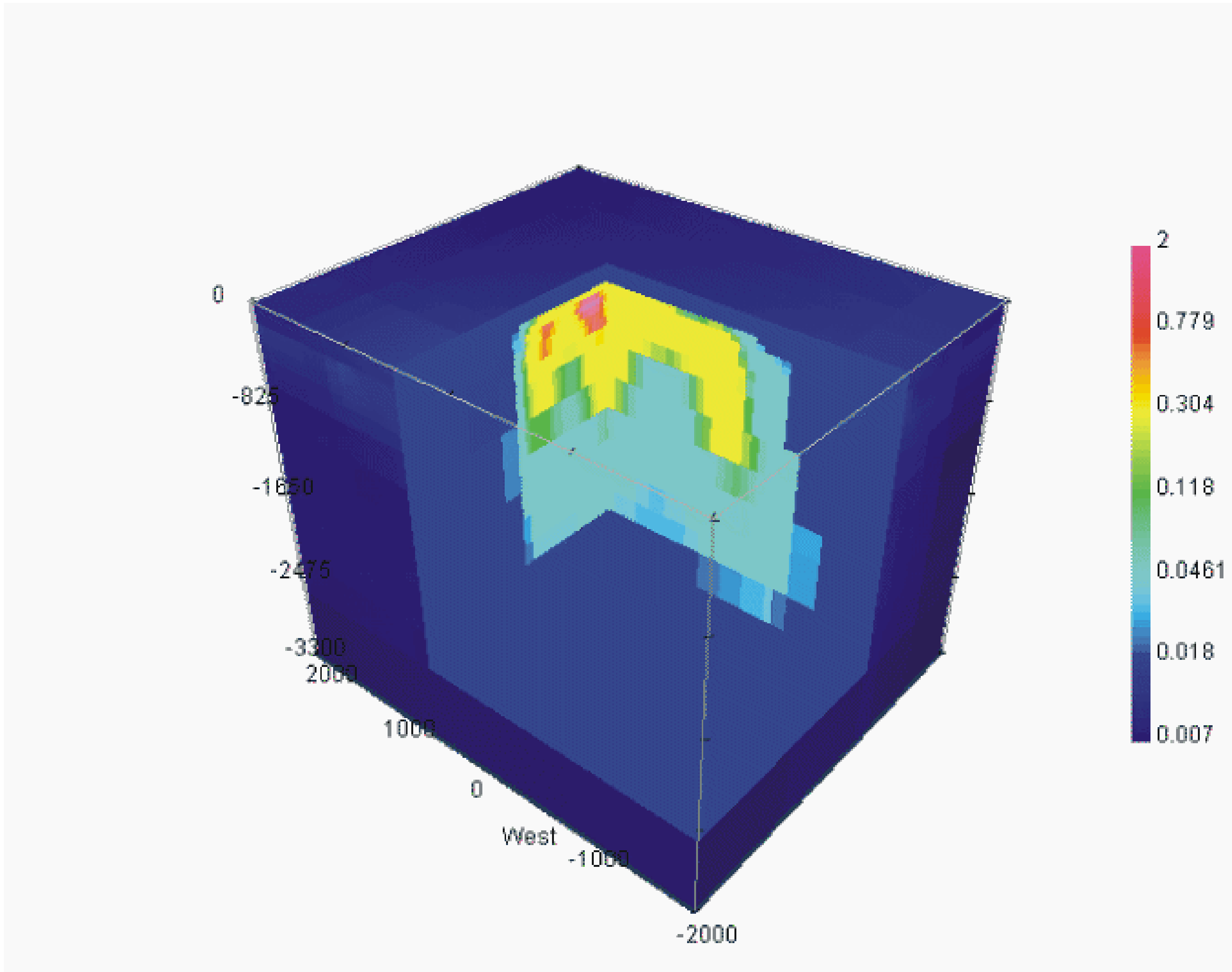


Inversion example: constructed model



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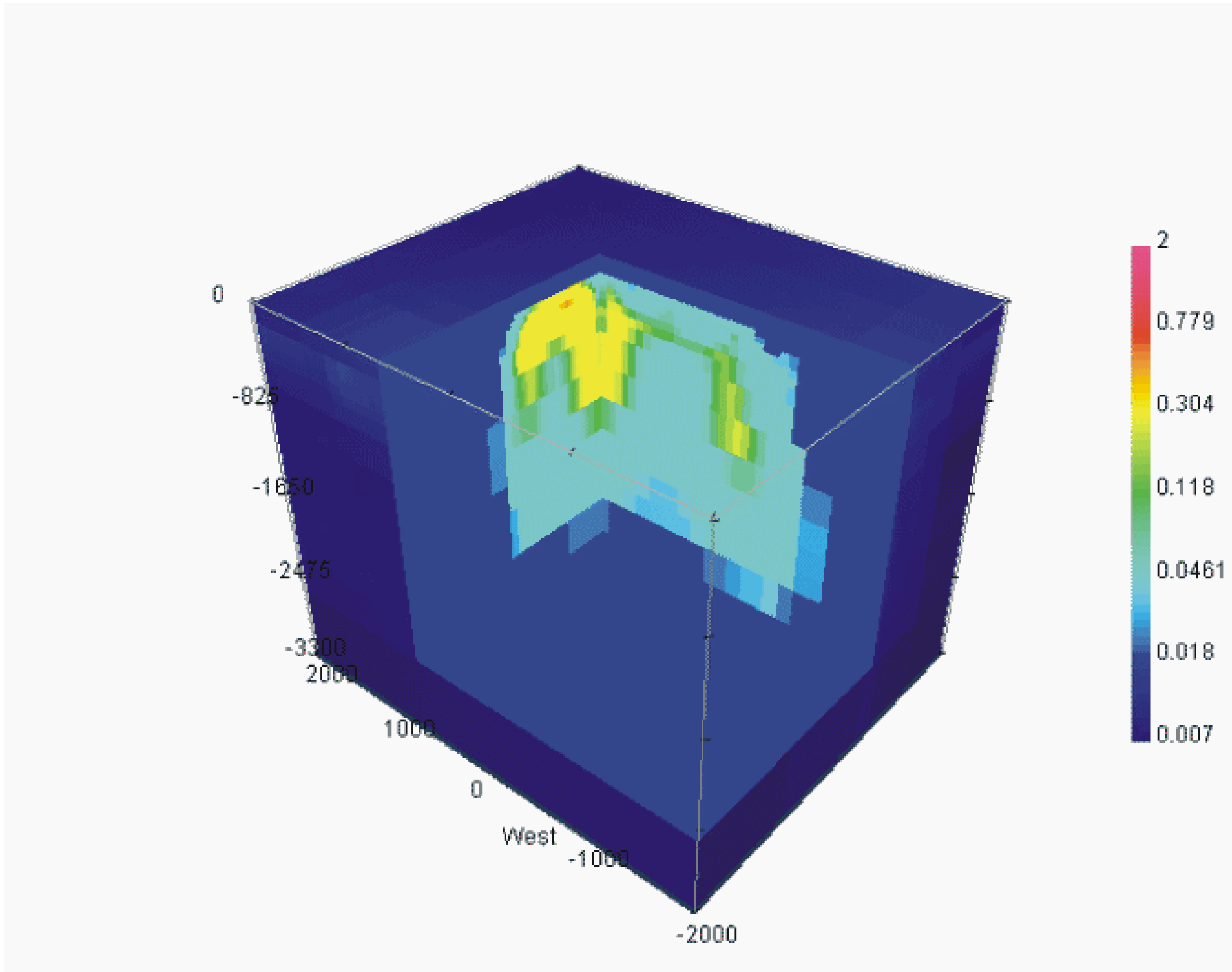

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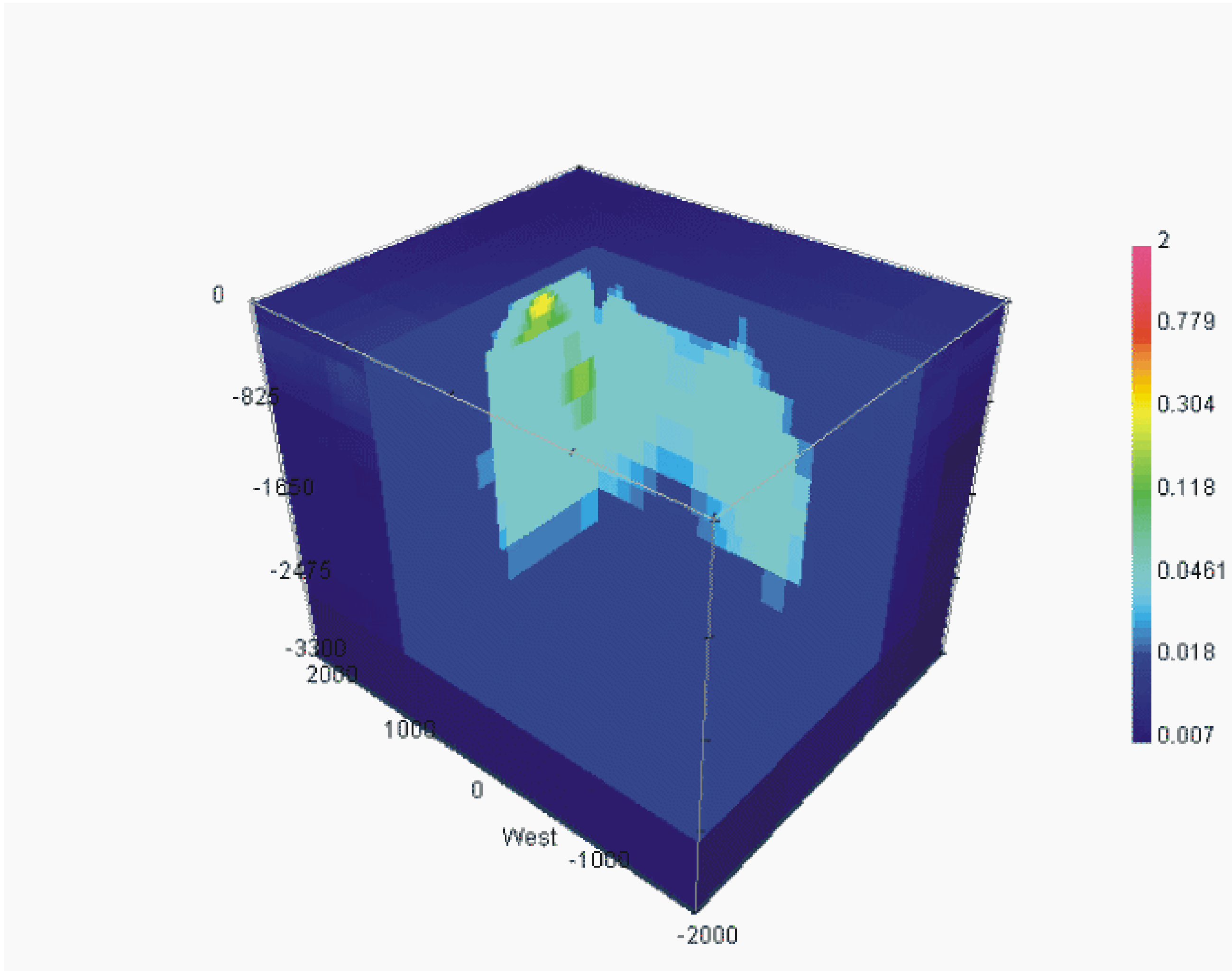
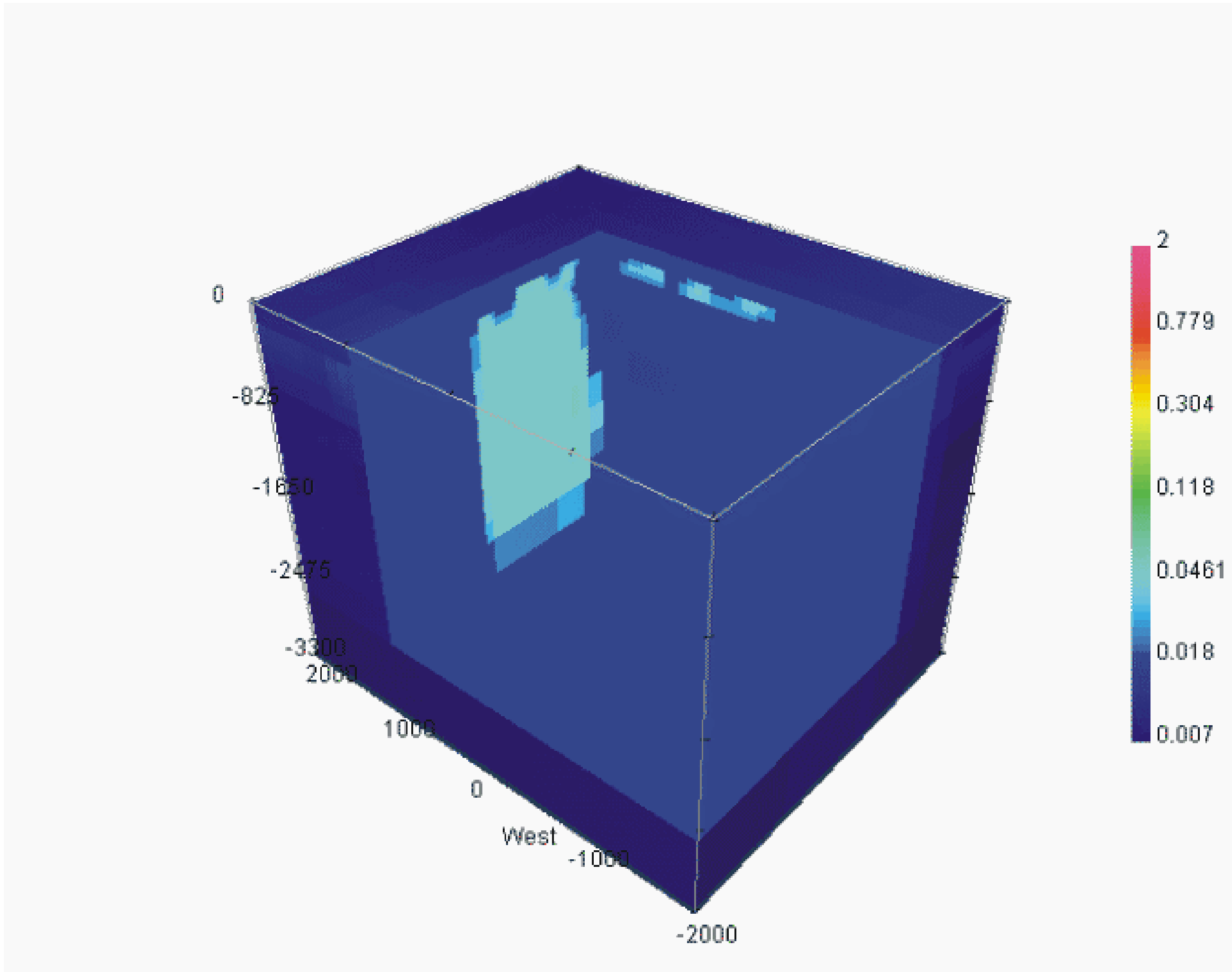
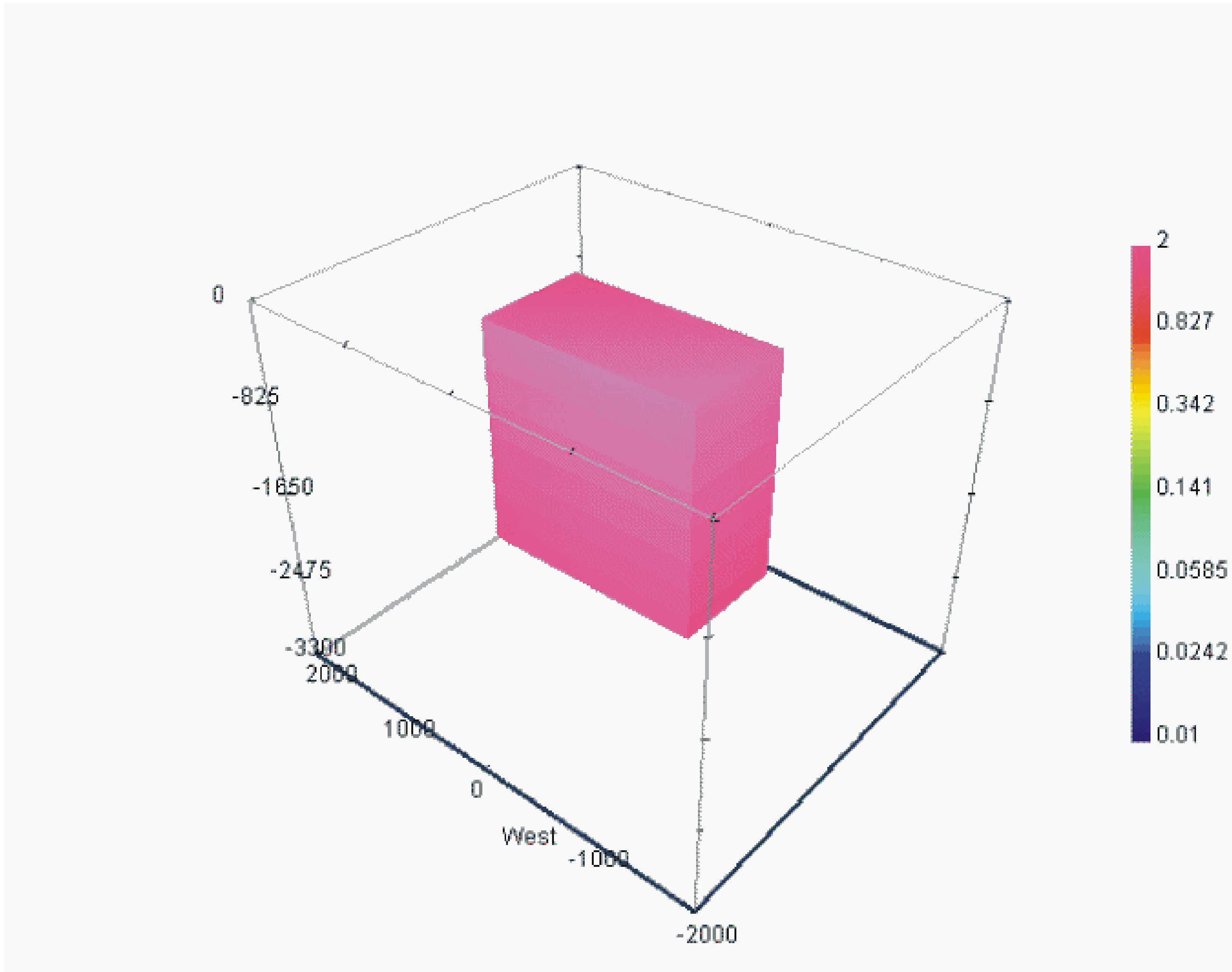


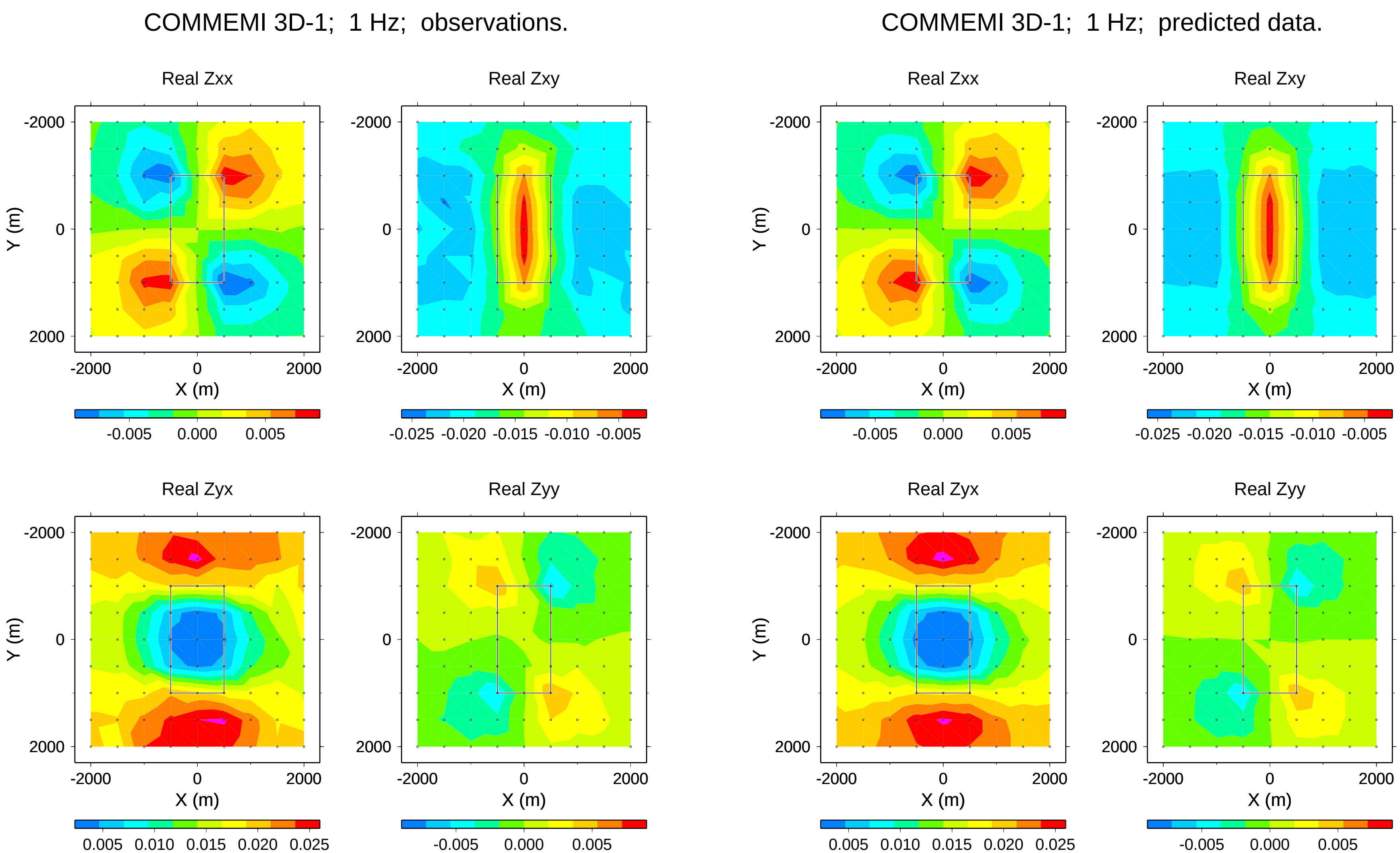
Inversion example: constructed model



Inversion example: true model

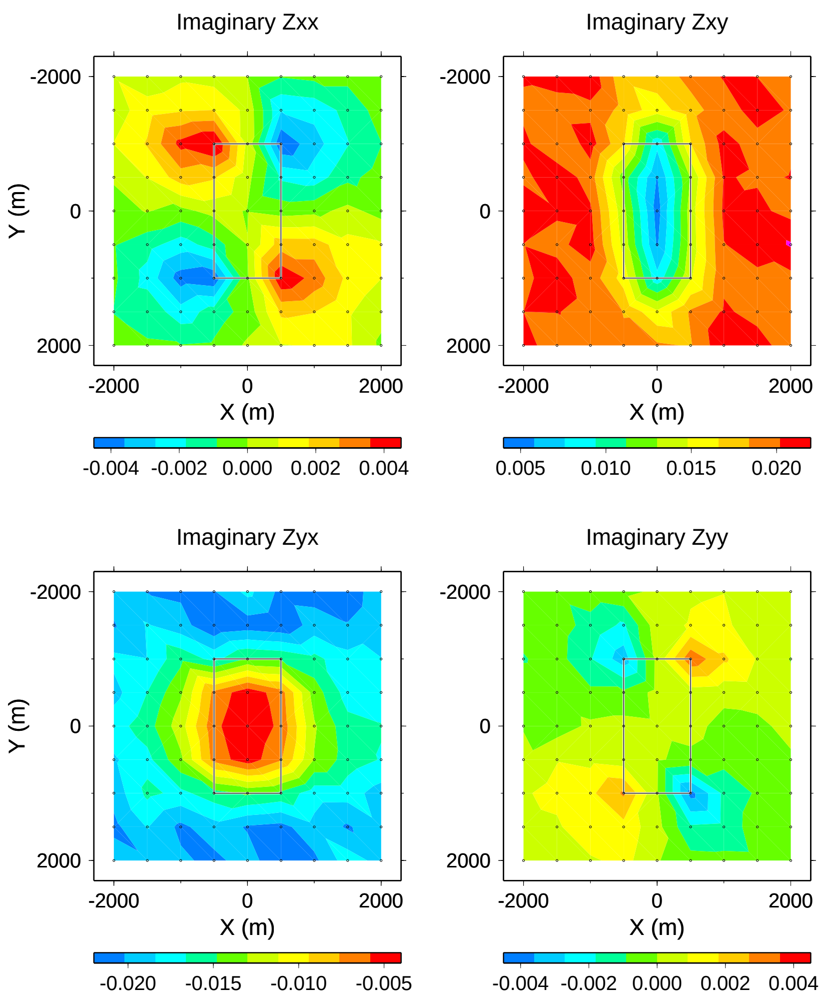


Inversion example: observed & predicted data

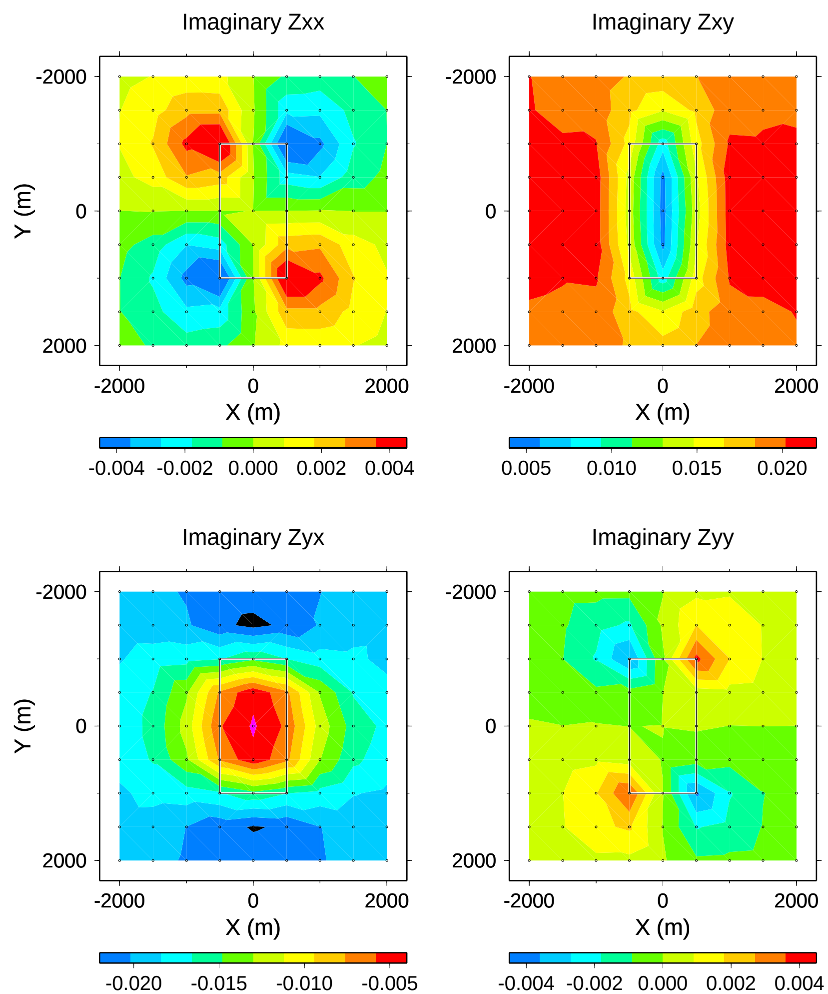


Inversion example: observed & predicted data

COMMEMI 3D-1; 1 Hz; observations.



COMMEMI 3D-1; 1 Hz; predicted data.



Inversion: example

- Additional weighting in the immediate near-surface.
- Final misfit: 3600.
- Number of values of β: ~ 6;
  - number of Gauss-Newton iterations per β: ~ 4;
  - number of IPCG iterations per G-N iteration: ~ 10;
  - "forward modellings" for $\mathbf{J}$ & $\mathbf{J}^T$ operations: 4.

$\rightarrow$ Computation time: $\sim$5 days.

Summary

- ⋆ Efficient, robust forward-modelling algorithm.
- ⋆ Three-dimensional, iterative, linearised, minimum-structure inversion procedure.
- ⋆ Iterative solution of the system of equations.
- ⋆ Application of Jacobian matrix done using only sparse matrix-vector operations.