Gauged Vector Finite-Element Schemes for the Geophysical EM problem for Unique Potentials and Fields

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Outline

The Forward problem

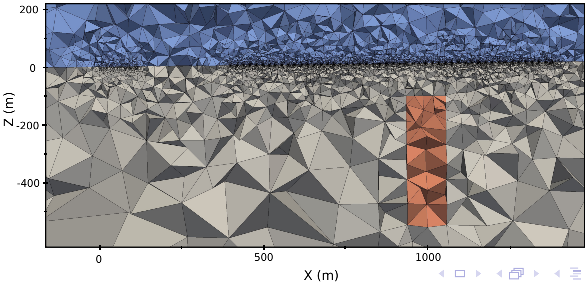
Calculating the electric field using the Helmholtz equation, E-filed system

$$\nabla \times \nabla \times \mathbf{E} + i\omega\mu\sigma\mathbf{E} - \omega^2\mu\epsilon\mathbf{E} = -i\omega\mu\mathbf{J}_e^s - \nabla \times \mathbf{J}_m^s \qquad (1)$$

$$\mathbf{n} \times \mathbf{E} = \mathbf{0}$$

$\mathbf{J}_e^s$ and $\mathbf{J}_m^s$ are electric and magnetic source current densities.

Minimizing equation 1 over the physical domain Ω



Discretization

Method of weighted residuals

$$\mathbf{R} = \int_{\Omega} \mathbf{W} \cdot \mathbf{r} \; d\Omega \tag{2}$$

$\mathbf{r}$ is the residual function.

Finite-element basis functions

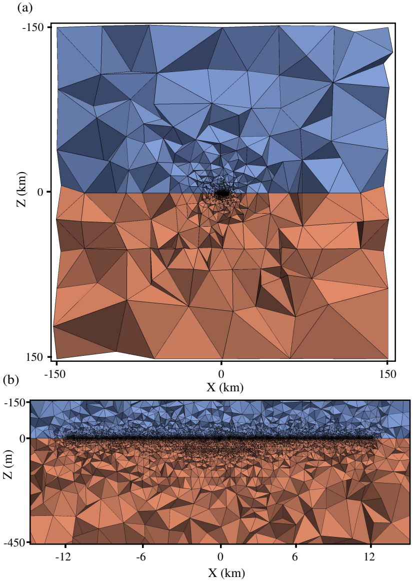
$$\tilde{\mathbf{E}} = \sum_{i=1}^{N_{edges}} \tilde{E}_i \; \mathbf{N}_i \tag{3}$$

$\mathbf{N}_i$ linear edge elements.

$$\int_{\Omega} (\nabla \times \mathbf{W}) \cdot (\nabla \times \mathbf{N}_i) \; d\Omega + \int_{\Omega} (ik_1 - k_2)(\mathbf{W} \cdot \mathbf{N}_i) \; d\Omega = \int_{\Omega} \mathbf{W} \cdot \mathbf{S} \; d\Omega \tag{4}$$

$$\mathbf{S}(\mathbf{r}) + (ik_1 - k_2)\mathbf{M}(\sigma, \mathbf{r}) = \mathbf{RHS} \tag{5}$$

E-dipole source, frequency 0.1 Hz, half space of 0.01 S/m



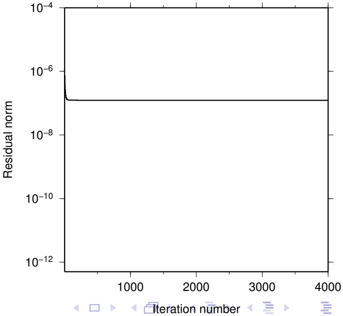
ill-conditioned system

$$[\mathbf{S}(\mathbf{r}) + (ik_1 - k_2)\mathbf{M}(\sigma, \mathbf{r})]\ \tilde{\mathbf{E}} = \mathbf{RHS}$$

For small $ik_1 - k_2$: nearly singular **LHS**

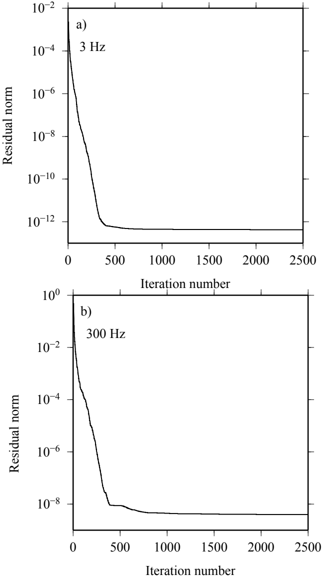
Non-smooth **RHS**

Iterative solver GMRES with ILU preconditioning

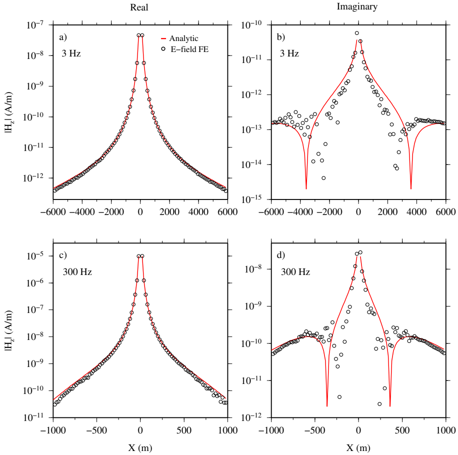


Another example

H-dipole source, half space of 0.01 S/m



The residual norm is not small enough to give the correct field



A-ϕ system: Remedy for the slow iterative problem

$$\tilde{\mathbf{E}} = -i\omega\tilde{\mathbf{A}} - \nabla\tilde{\phi} \tag{6}$$

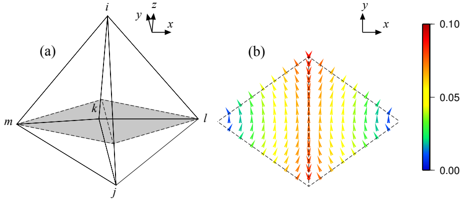
The induction equation

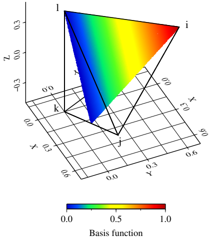
$$\nabla \times \nabla \times \tilde{\mathbf{A}} + (i\omega\mu\sigma + \omega^2\mu\epsilon)\tilde{\mathbf{A}} + (\mu\sigma + i\omega\mu\epsilon)\nabla\tilde{\phi} = \mu\mathbf{J}^s \tag{7}$$

Equation of conservation of charge

$$-i\omega\nabla\cdot(\sigma\tilde{\mathbf{A}}) - \nabla\cdot(\sigma\nabla\tilde{\phi}) + \omega^2\nabla\cdot(\epsilon\tilde{\mathbf{A}}) - i\omega\nabla\cdot(\epsilon\nabla\tilde{\phi}) = -\nabla\cdot\mathbf{J}^s \tag{8}$$

Finite-element approximation of the potentials





$$\tilde{\mathbf{A}} = \sum_{i=1}^{N_{edges}} \tilde{A}_i \, \mathbf{N}_i$$

$$\tilde{\phi} = \sum_{k=1}^{N_{nodes}} \tilde{\phi} \, N_k$$

System to solve

$$\begin{pmatrix} \mathbf{S} + i\omega\mu\mathbf{M}_1 + \omega^2\mu\mathbf{M}_2 & \mu\mathbf{F}_1 + i\omega\mu\mathbf{F}_2 \\ i\omega\mathbf{F}_1^T + \omega^2\mathbf{F}_2^T & \mathbf{H}_1 + i\omega\mathbf{H}_2 \end{pmatrix} \begin{pmatrix} \tilde{A} \\ \tilde{\phi} \end{pmatrix} = \begin{pmatrix} \mu_0 S_1 \\ S_2 \end{pmatrix}, \quad (9)$$

$$\mathbf{S} = \int_\Omega \nabla \times \mathbf{N}_i \cdot \nabla \times \mathbf{N}_j \; d\Omega$$

$$\mathbf{M}_2 = \int_\Omega \epsilon \mathbf{N}_i \cdot \mathbf{N}_j \; d\Omega$$

$$\mathbf{F}_2 = \int_\Omega \epsilon \mathbf{N}_i \cdot \nabla N_k \; d\Omega$$

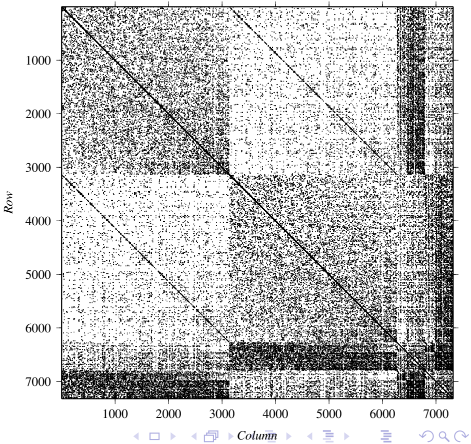
$$\mathbf{H}_1 = \int_\Omega \sigma \nabla N_k \cdot \nabla N_l \; d\Omega$$

$$\mathbf{M}_1 = \int_\Omega \sigma \mathbf{N}_i \cdot \mathbf{N}_j \; d\Omega$$

$$\mathbf{F}_1 = \int_\Omega \sigma \mathbf{N}_i \cdot \nabla N_k \; d\Omega$$

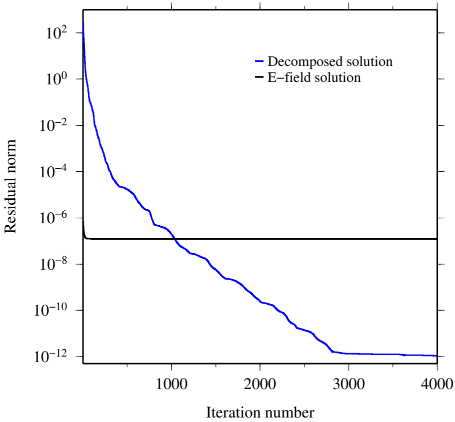
$$\mathbf{D} = -\int_\Omega \nabla N_k \cdot \mathbf{N}_j \; d\Omega$$

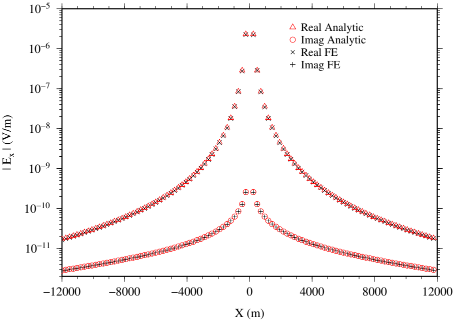
$$\mathbf{H}_2 = \int_\Omega \epsilon \nabla N_k \cdot \nabla N_l \; d\Omega$$



Fast convergence for A-ϕ system

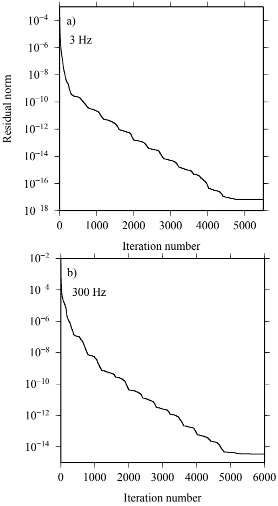
E-dipole and half space

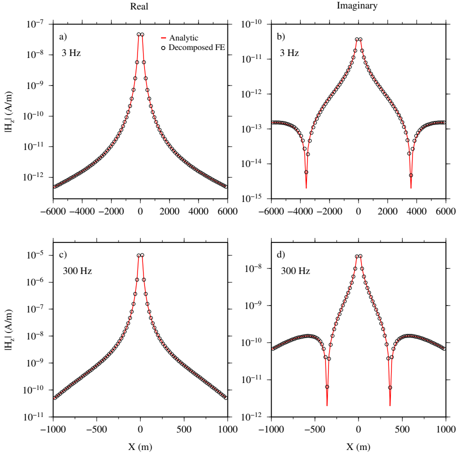




Fast convergence for A-ϕ system

H-dipole and half space





Uniqueness problem

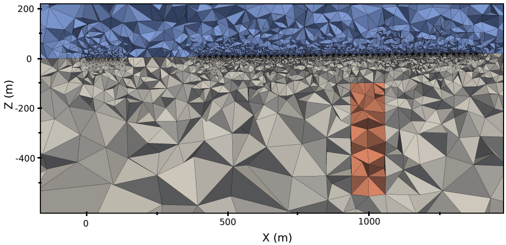
The ungauged system

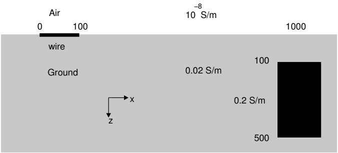
$$\nabla \times \nabla \times \tilde{\mathbf{A}} + (i\omega\mu\sigma + \omega^2\mu\epsilon)\tilde{\mathbf{A}} + (\mu\sigma + i\omega\mu\epsilon)\nabla\tilde{\phi} = \mu\mathbf{J}^s \quad (10)$$

$$-i\omega\nabla\cdot(\sigma\tilde{\mathbf{A}}) - \nabla\cdot(\sigma\nabla\tilde{\phi}) + \omega^2\nabla\cdot(\epsilon\tilde{\mathbf{A}}) - i\omega\nabla\cdot(\epsilon\nabla\tilde{\phi}) = -\nabla\cdot\mathbf{J}^s \quad (11)$$

$$\tilde{\mathbf{E}} = -i\omega\tilde{\mathbf{A}} - \nabla\tilde{\phi} \quad (12)$$

Grounded wire and conductive prism example, Frequency 3 Hz

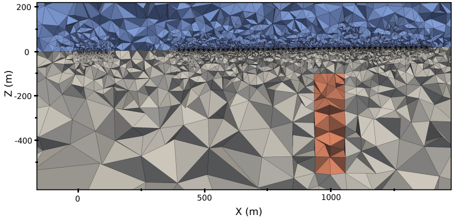




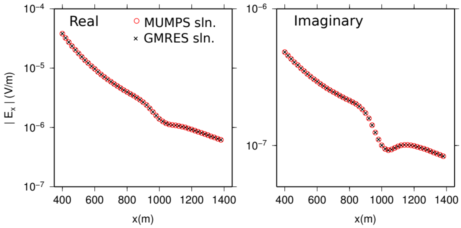
Iterative solution, GMRES from SPARSKIT (Saad, 1990)

Direct solution, MUMPS (Amestoy et al., 2001)

The ungauged system produces unique E and H

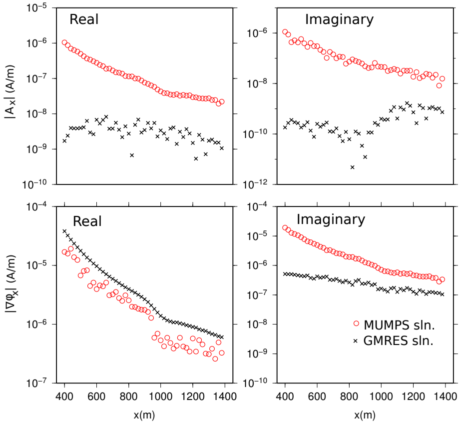


Total fields



But non-unique A and ϕ

Calculated Potentials



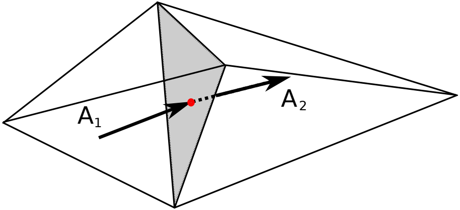
Continuity study for the source of non-uniqueness

$$\tilde{\mathbf{A}} = \sum_{i=1}^{N_{edges}} \mathbf{N}_i$$

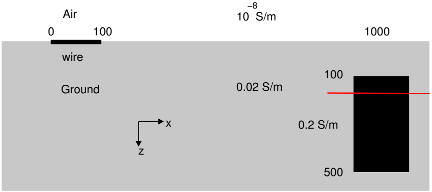
$$\nabla \cdot \mathbf{N} = 0.$$

$$\nabla \cdot \mathbf{A}|_{\partial\Omega} \neq 0.$$

$$\mathbf{A}_1 \cdot \mathbf{n}_1 = \mathbf{A}_2 \cdot \mathbf{n}_2 \quad ??$$

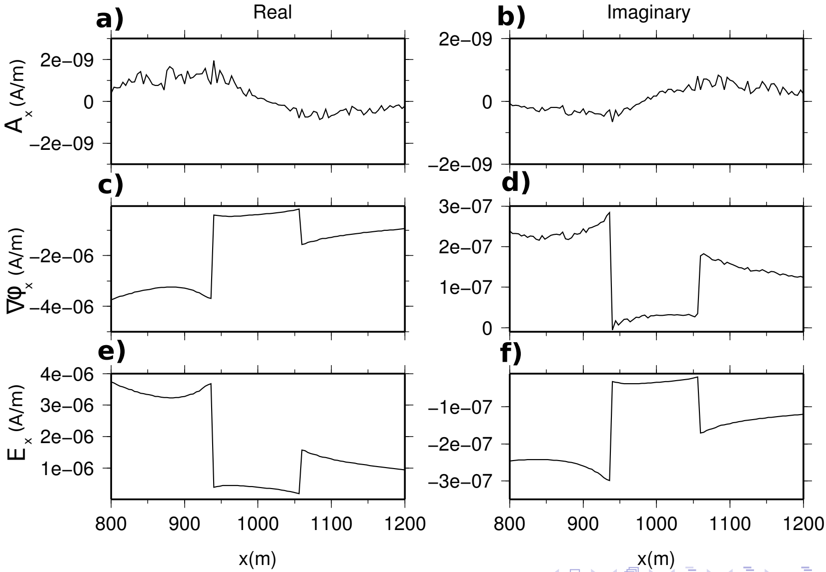


Data for a profile at z=120.

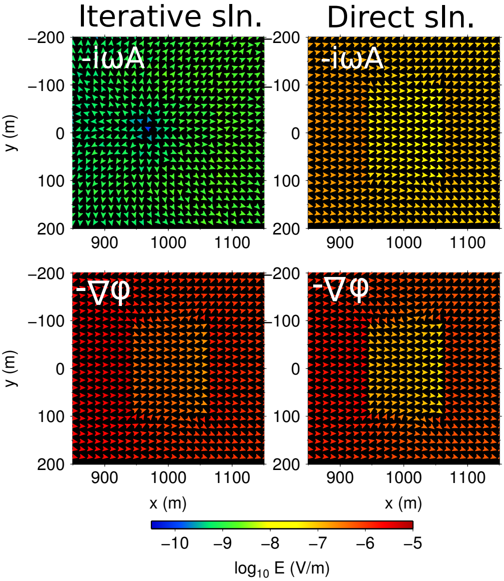


Continuity study for the A-ϕ ungauged system

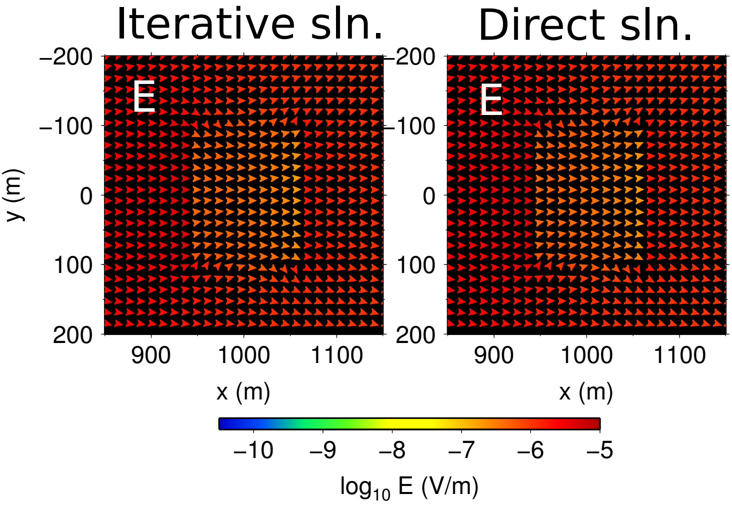
Noisy **A**



Leakage in A causes non-unique A and ϕ



Total E is unique



Gauge fixing

Conventional method does not work!

$$\nabla \times \nabla \times \tilde{\mathbf{A}} - \nabla(\nabla \cdot \tilde{\mathbf{A}}) + (i\omega\mu\sigma + \omega^2\mu\epsilon)\tilde{\mathbf{A}} + (\mu\sigma + i\omega\mu\epsilon)\nabla\tilde{\phi} = \mu\mathbf{J}^s$$

$$\int_\Omega \mathbf{N}_i \cdot \nabla(\nabla \cdot \tilde{\mathbf{A}})\ d\Omega = 0.$$

Method 1: explicit gauging

$$\nabla \times \nabla \times \tilde{\mathbf{A}} + (i\omega\mu\sigma + \omega^2\mu\epsilon)\tilde{\mathbf{A}} + (\mu\sigma + i\omega\mu\epsilon)\nabla\tilde{\phi} = \mu\mathbf{J}^s$$
$$-i\omega\nabla \cdot (\sigma\tilde{\mathbf{A}}) + \nabla \cdot \tilde{\mathbf{A}} - \nabla \cdot (\sigma\nabla\tilde{\phi}) + \omega^2\nabla \cdot (\epsilon\tilde{\mathbf{A}}) - i\omega\nabla \cdot (\epsilon\nabla\tilde{\phi}) = -\nabla \cdot \mathbf{J}^s$$

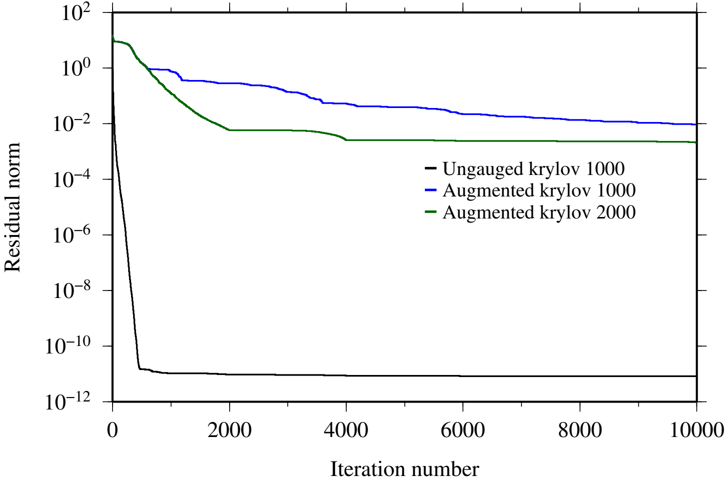
$$\begin{pmatrix} \mathbf{S} + i\omega\mu\mathbf{M}_1 + \omega^2\mu\mathbf{M}_2 & \mu\mathbf{F}_1 + i\omega\mu\mathbf{F}_2 \\ i\omega\mathbf{F}_1^T + \omega^2\mathbf{F}_2^T + \mathbf{D} & \mathbf{H}_1 + i\omega\mathbf{H}_2 \end{pmatrix} \begin{pmatrix} \tilde{A} \\ \tilde{\phi} \end{pmatrix} = \begin{pmatrix} \mu_0 S_1 \\ S_2 \end{pmatrix}, \quad (13)$$

$$\mathbf{D} = \int_\Omega N_k(\nabla \cdot \mathbf{N}_i)\ d\Omega$$

Solutions - Gauged system

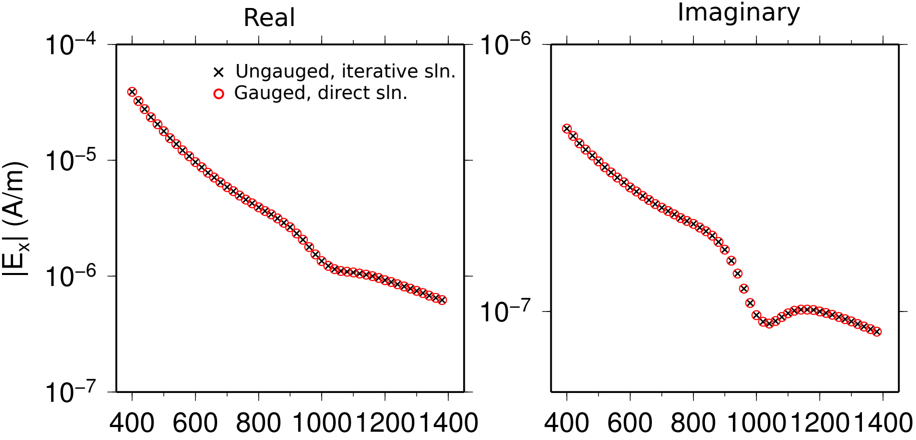
Grounded wire and prism example

113611 cells, 18669 nodes and 132507 edges.
Iterative solver, GMRES is slow even for large Krylov subspaces

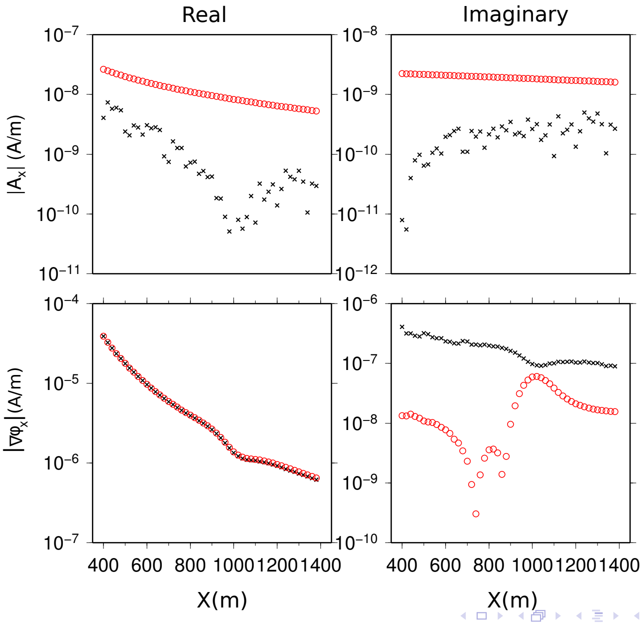


Direct solver, MUMPS is used.

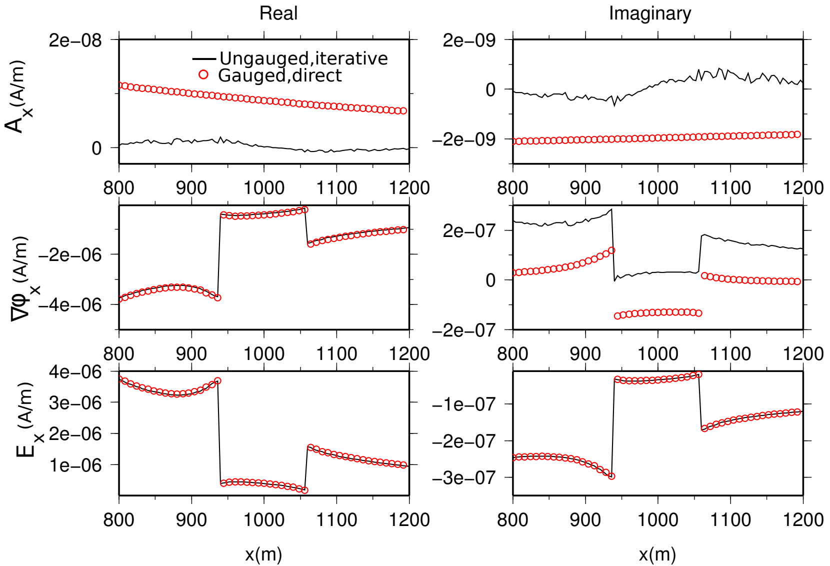
Calculated Electric field



Calculated Potentials



Discontinuity fix, smooth vector potential



Another method for gauging

Method 2: Applying the Coulomb gauge condition directly and using a Lagrange multiplier in the induction equation

$$\nabla \times \nabla \times \mathbf{A} + (i\omega\mu\sigma + \omega^2\mu\epsilon)\mathbf{A} + (\mu\sigma + i\omega\mu\epsilon)\nabla\phi + \nabla\lambda = \mu\mathbf{J}^s \quad (14)$$

$$-i\omega\nabla\cdot(\sigma\mathbf{A}) - \nabla\cdot(\sigma\nabla\phi) + \omega^2\epsilon\nabla\cdot\mathbf{A} - i\omega\epsilon\nabla\cdot\nabla\phi = -\nabla\cdot\mathbf{J}^s \quad (15)$$

$$-\nabla\cdot\mathbf{A} = 0 \quad (16)$$

Finite-element approximation

Nodal elements for the Lagrange multiplier

$$\tilde{\lambda} = \sum_{i=1}^{N_{nodes}} \tilde{\lambda}_i N_i \quad (17)$$

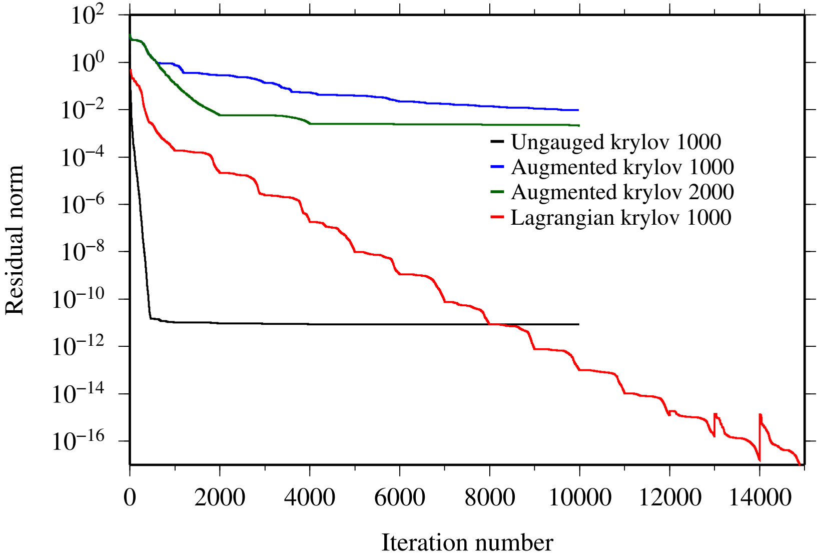
The system to solve

$$\begin{pmatrix} \mathbf{S} + i\omega\mu\mathbf{M}_1 + \omega^2\mu\mathbf{M}_2 & \mu\mathbf{F}_1 + i\omega\mu\mathbf{F}_2 & \mathbf{L} \\ i\omega\mathbf{F}_1^T + \omega^2\mathbf{F}_2^T & \mathbf{H}_1 + i\omega\mathbf{H}_2 & 0 \\ \mathbf{L}^T & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{A} \\ \tilde{\phi} \\ \tilde{\lambda} \end{pmatrix} = \begin{pmatrix} \mu_0 S_1 \\ S_2 \\ 0 \end{pmatrix}, \tag{18}$$

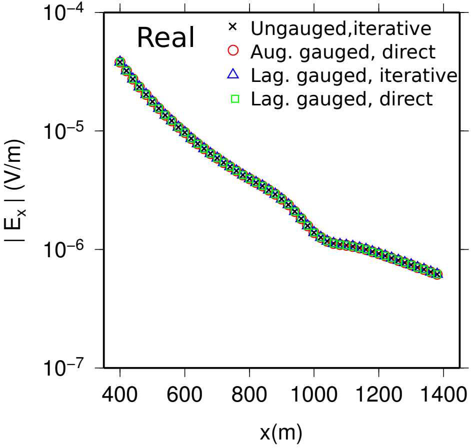
$$\mathbf{L} = \int_\Omega \mathbf{N}_i \cdot \nabla\lambda_k \; d\Omega$$

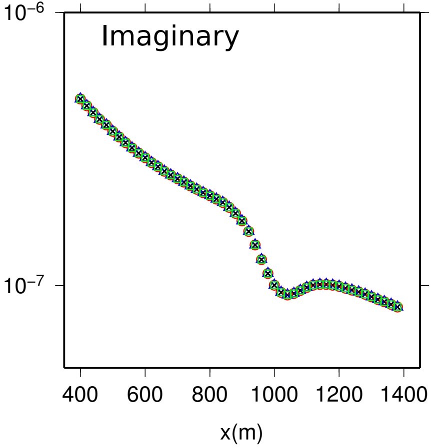
Lagrange multiplier system

A better iterative solution

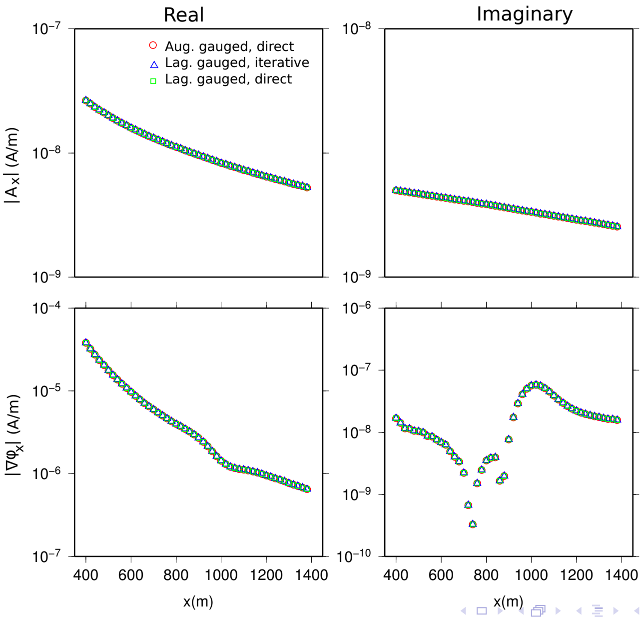


Calculated Fields

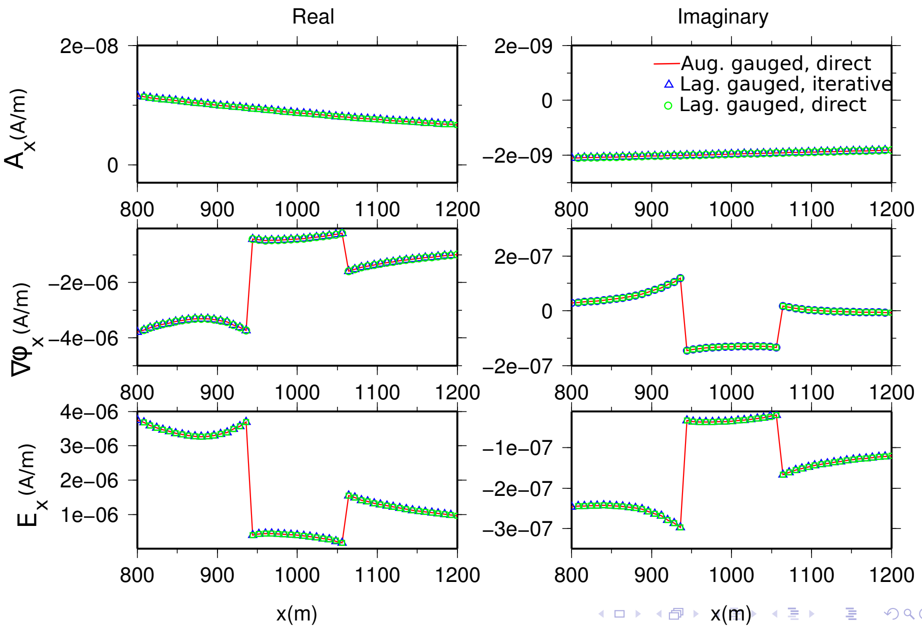




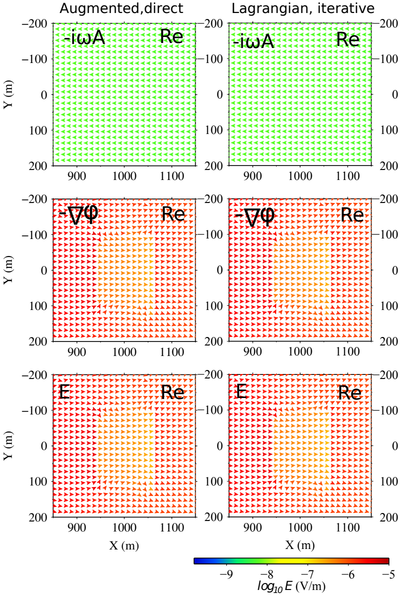
Calculated potentials



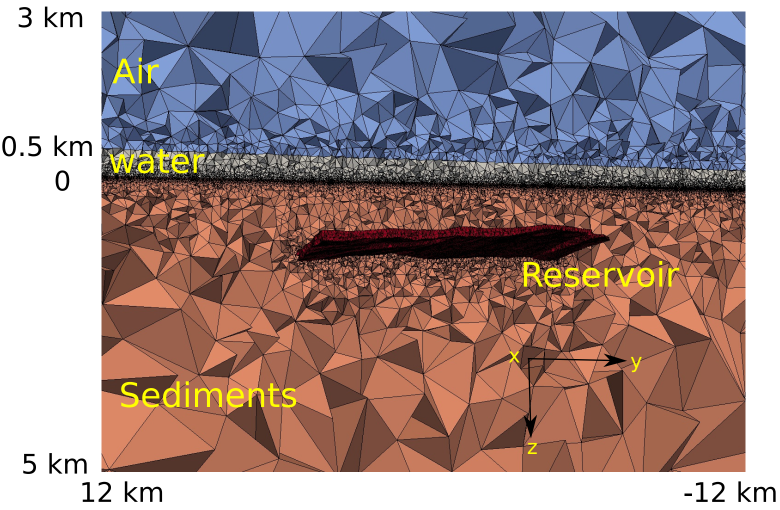
Discontinuity fix



Unique fields

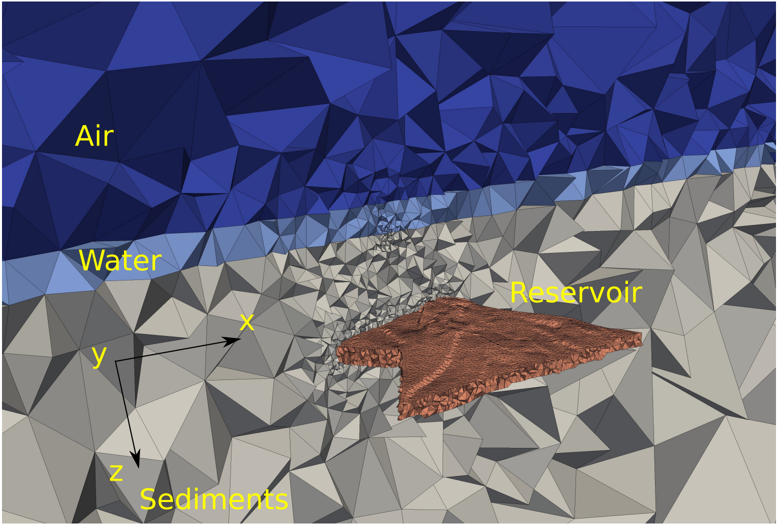


A realistic 3D model offshore Newfoundland, Canada - topographic reservoir is at $z \approx 1100$ m



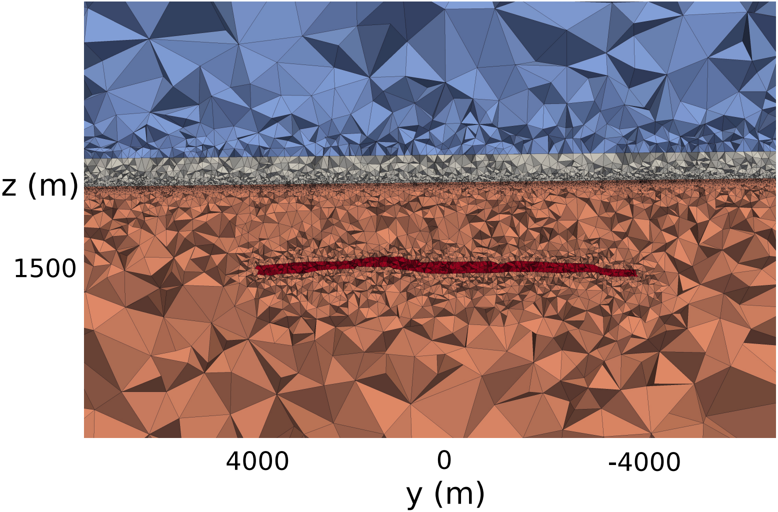
Courtesy of Husky Energy

Another perspective



Courtesy of Husky Energy

A realistic 3D model- yz cross section



Courtesy of Husky Energy

Physical properties

Layer	conductive (S/m)	Relative Permittivity (F/m)	Permeability (H/m)
Air	10^{-8}	1	μ_0
Sea water	3.3	80	μ_0
Sediments	0.71	1	μ_0
Reservoir	0.01	1	μ_0

Courtesy of Husky Energy

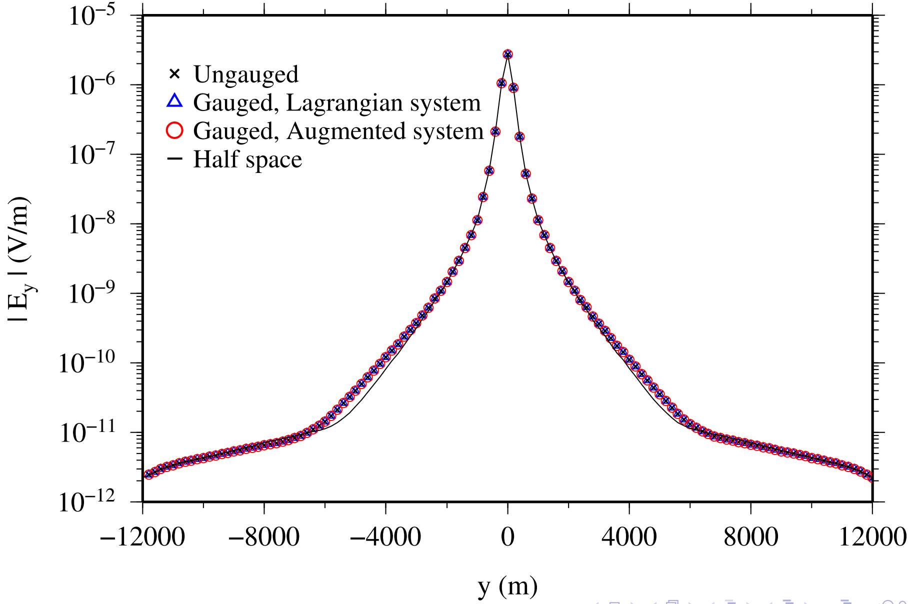
CSEM array

Electric dipole source of 200 m, Y-directed

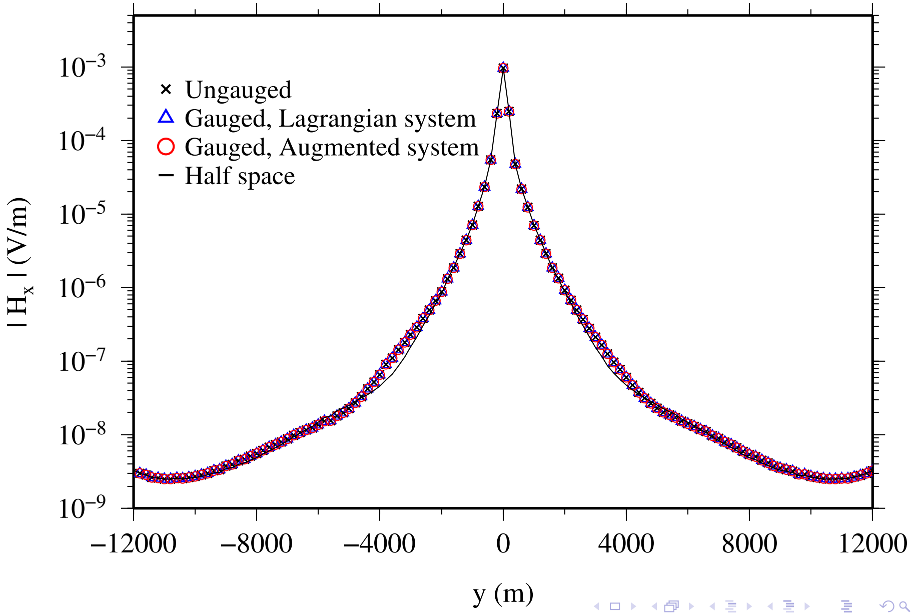
Observation locations Y-axis

Frequency 0.25 Hz

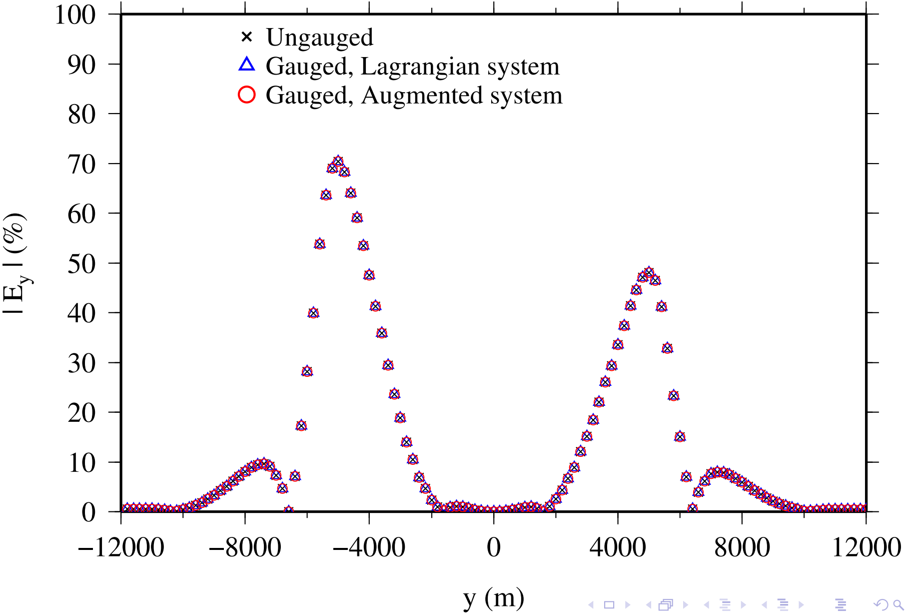
Electric field response



Magnetic field response



Secondary response, normalized



Solutions

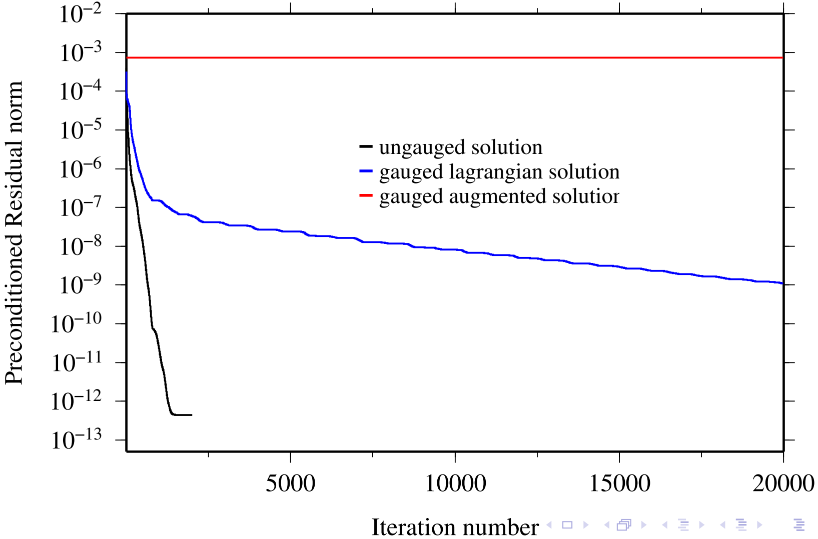
Computations:

System	cells	unknowns	nnz
Ungauged	643,477	1,710,090	98,163,278
Aug. Gauged	643,477	1,710,090	98,163,278
Lag. Gauged	643,477	1,920,406	119,477,836

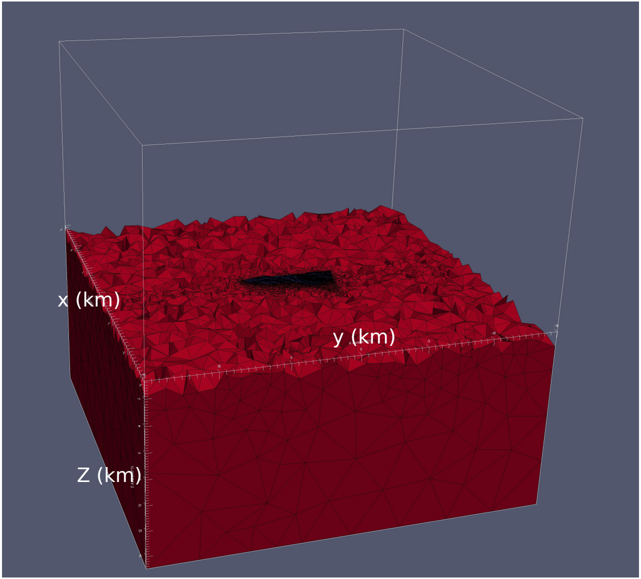
System	Krylov	Iterations	GMRES	time	MUMPS	time
Ungauged	800	2,000	15 GB	5868 sec	NA	NA
Aug. Gauged	800	-	15 GB	-	36 GB	1112 sec
Lag. Gauged	800	20,000	17 GB	76237 sec	50 GB	1779 sec

Solutions

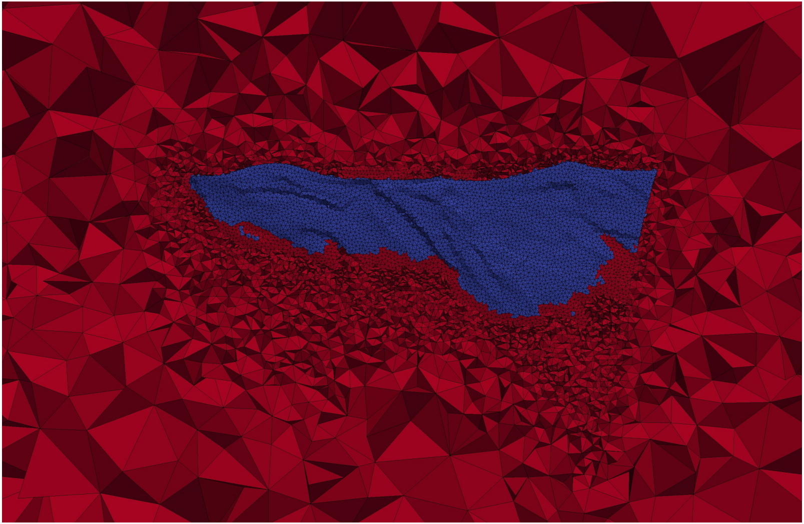
Iterative solution is slow for the gauged systems



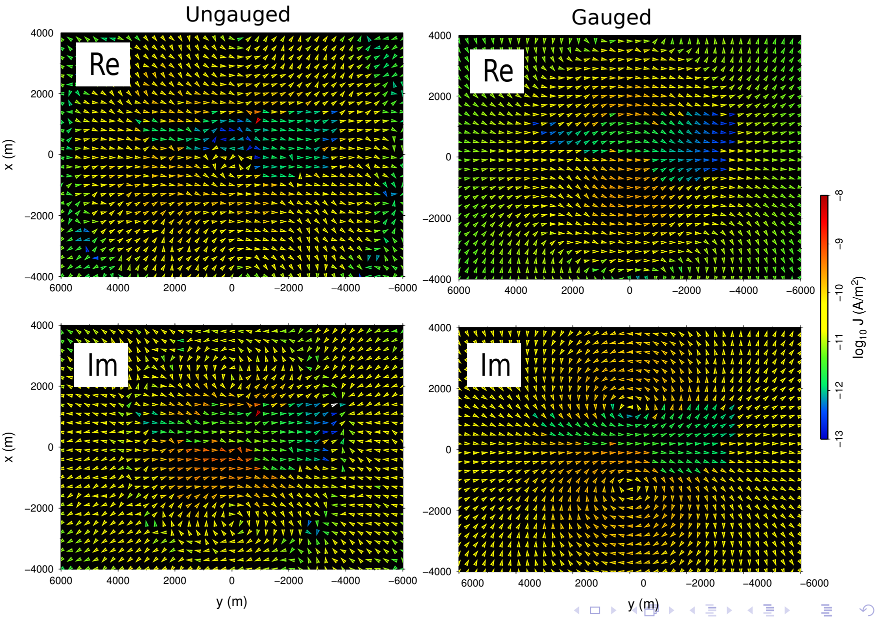
Inductive and Galvanic parts
Fields and currents at depth, xy cross section at $z = 1050$ m



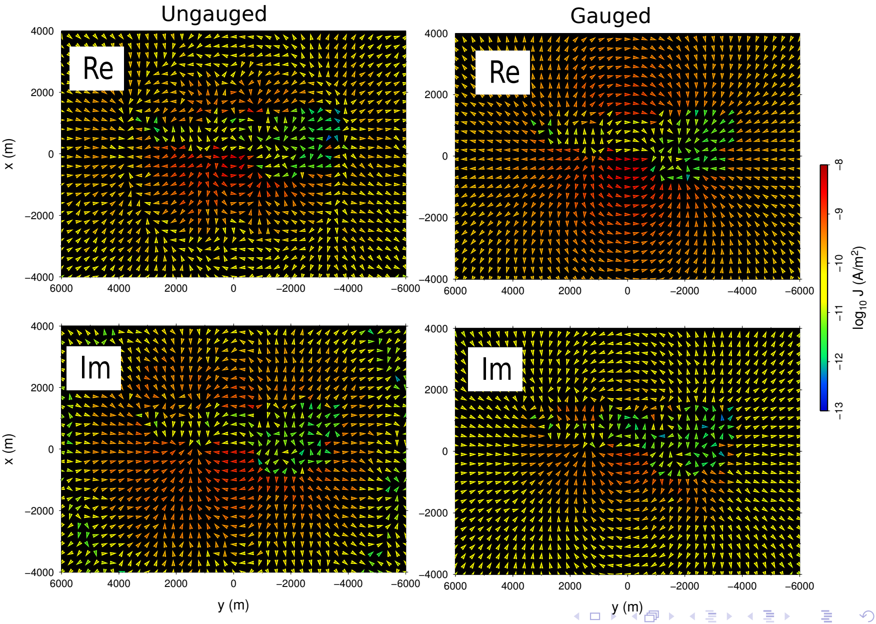
Close view at $z = 1050$ m



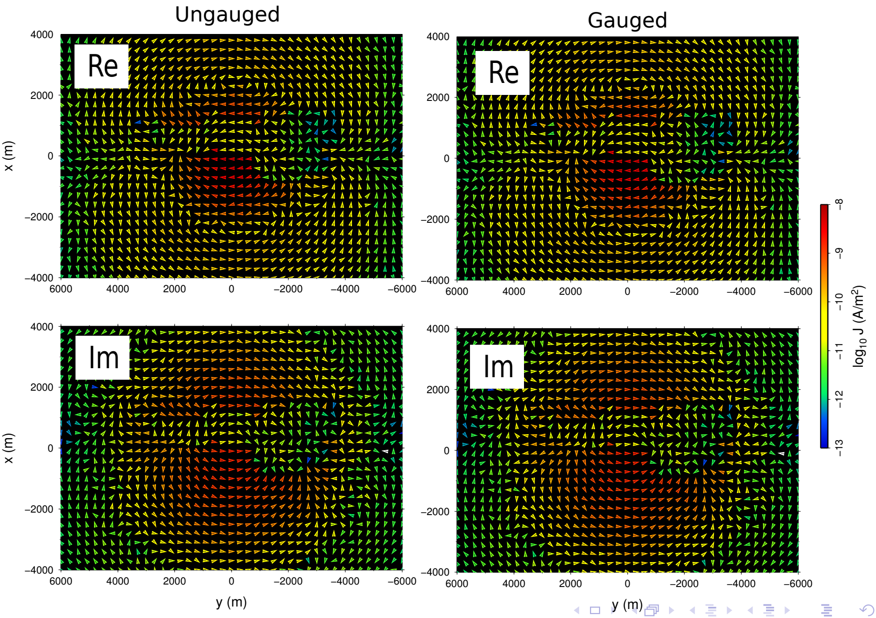
Inductive current, Gauged solution removed parasitic behavior



Galvanic current

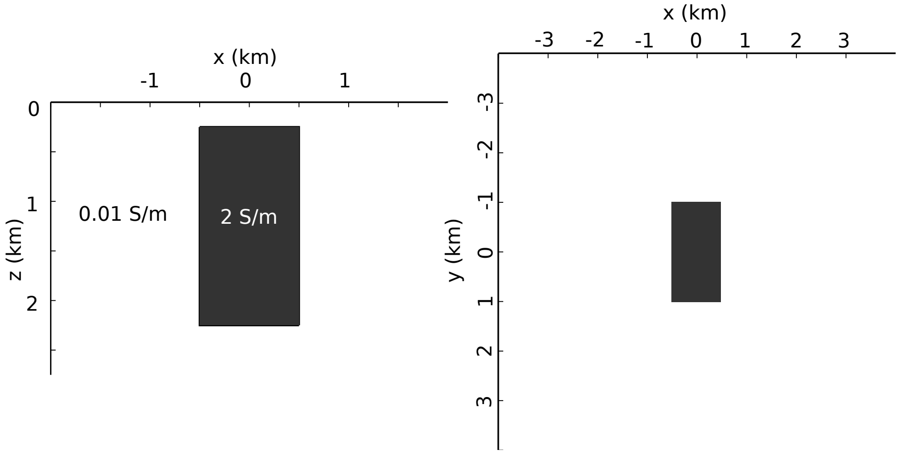


Total current density

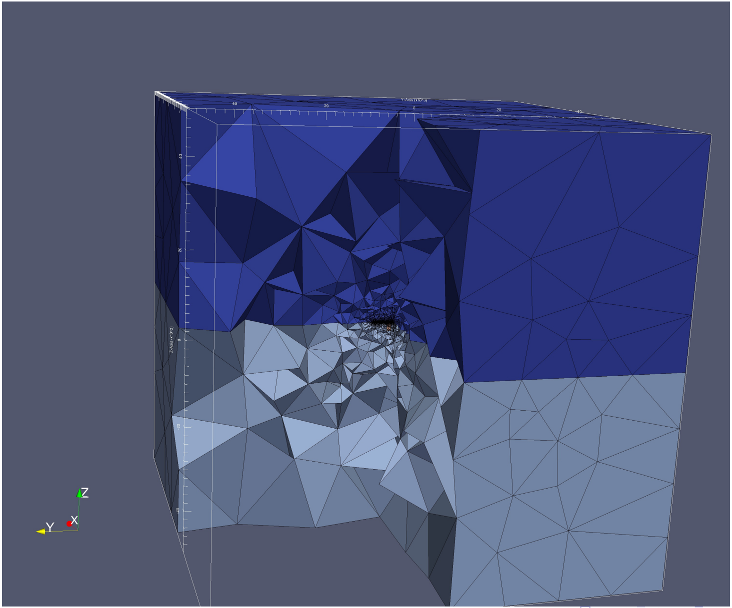


A Magnetotelluric example

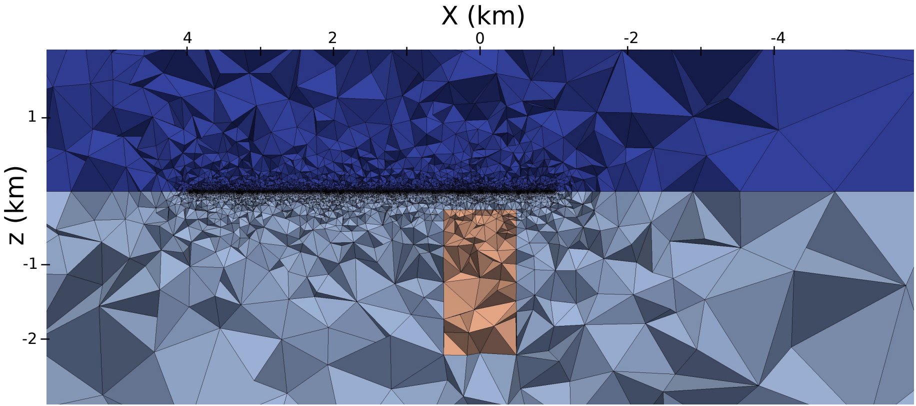
COMMEMI 3D-1A model



Meshed domain

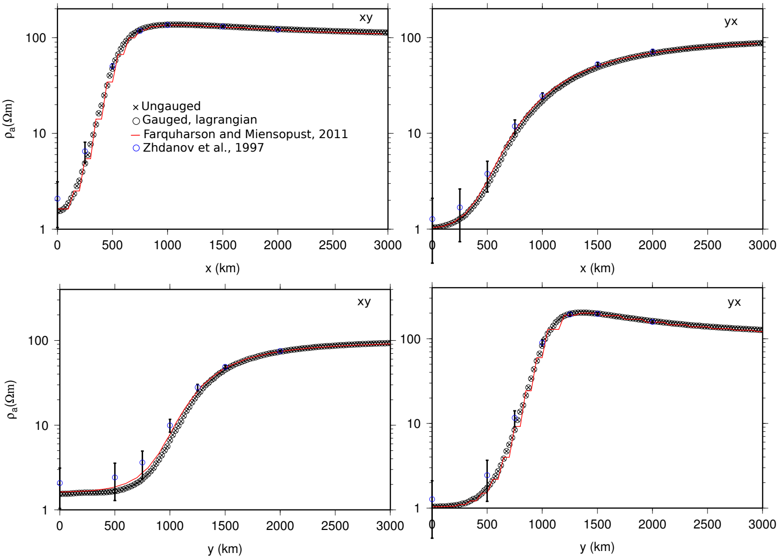


Closer view

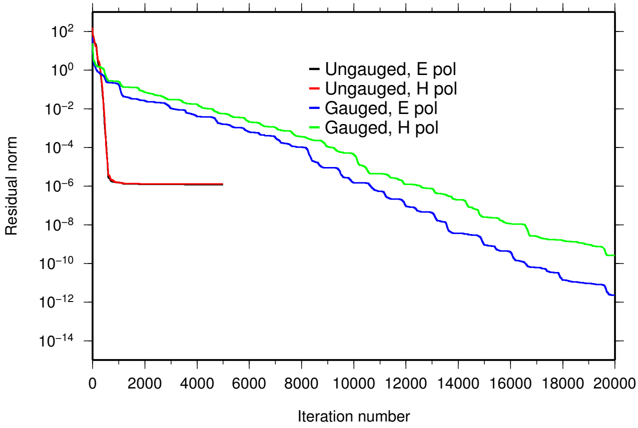


cells 526022, nodes 85752

Apparent resistivity for off-diagonal Z tensor

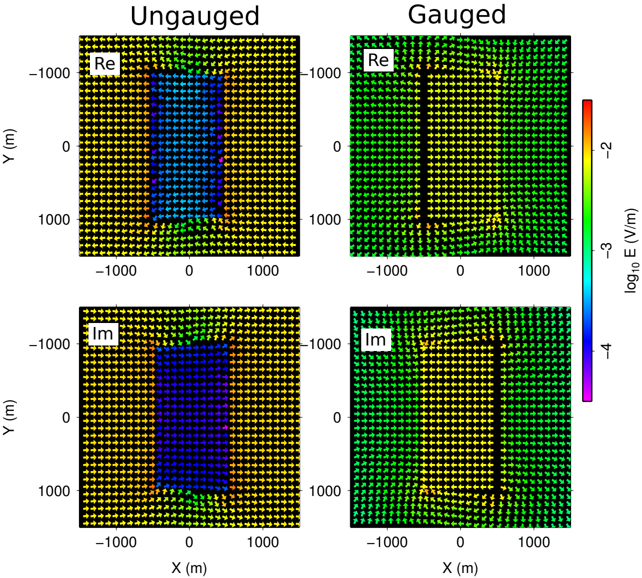


Convergence

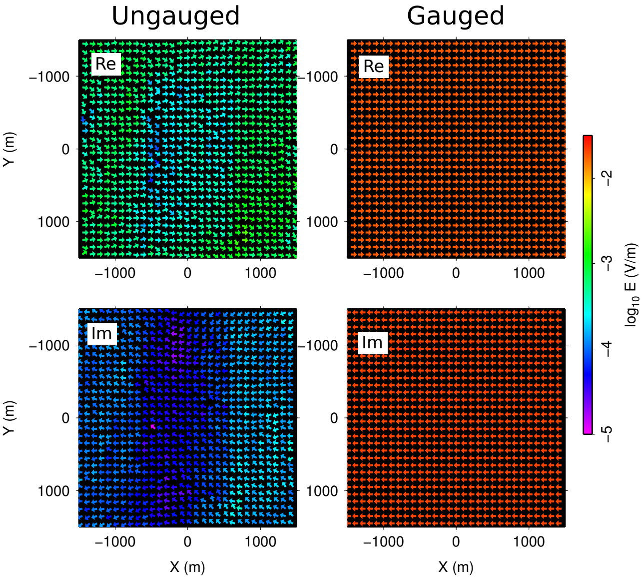


Method	nnz	dimension	iterative soln.	Direct soln.
Ungauged	80491966	1395528	11 GB	NA
Gauged	122457534	1967204	13 GB	Not needed

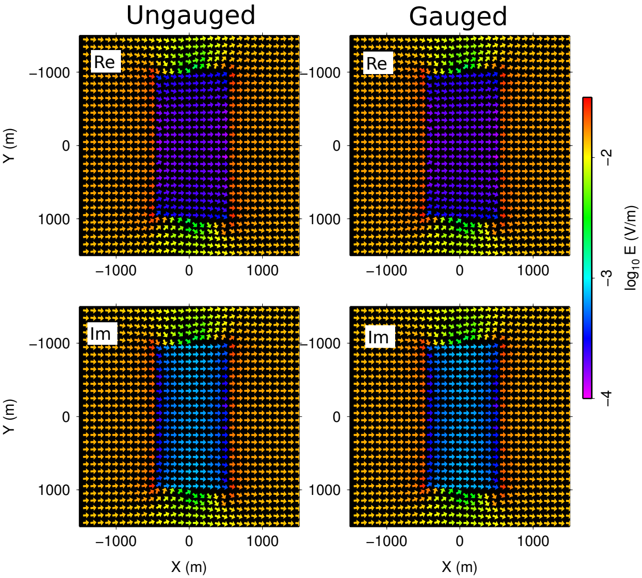
Inductive part



Galvanic part



Total field


Ungauged
Gauged
Re
Re
Im
Im
Y (m)
Y (m)
X (m)
X (m)
−1000
0
1000
log10 E (V/m)
−2
−3
−4

Numerical forward modeling methods for explicitly gauging the vectorial finite element solution of the geophysical electromagnetic problem are presented.

The electric field is decomposed into vector and scalar potentials and subsequently used in the Helmholtz diffusion equation and the equation of conservation of charge.

The nonuniqueness difficulty is counteracted by explicitly incorporating the Coulomb gauge condition in the system: firstly by augmenting the equation of conservation of charge and secondly by considering the gauge condition in an individual equation.