# Blocky models in minimum-structure inversions

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## Summary

The inclusion of diagonal spatial differences in an $l_1$-type measure of model structure enables piecewise-constant, blocky models with dipping interfaces to be constructed by minimum-structure inversion procedures. Examples are given for 2D MT inversion.

## Introduction

**Motivation**

Minimum-structure inversion procedures (that is, ones in which a measure of the amount of structure in the model is minimized in conjunction with a measure of data misfit, and in which the parameters being sought are the physical properties in the cells in an otherwise fixed mesh) are generally robust and reliable. However, the features in the models which are produced are typically smeared-out and fuzzy. The aim of the work presented here is to develop a minimum-structure inversion procedure that retains the robustness and reliability of existing implementations, but which can generate models comprising relatively uniform regions separated by sharp, distinct interfaces.

**Previous work**

A number of authors have successfully modified the traditional minimum-structure inversion procedure to use non-$l_2$-norm, non-sum-of-squares measures and thus produce models with sharp interfaces: Farquharson & Oldenburg (1998) for 1D EM; Portniaguine & Zhdanov (1999) for 3D gravity; Loke et al. (2003) and Farquharson & Oldenburg (2003) for 2D DC resistivity. However, it was found that the interfaces in the constructed models were predominantly horizontal or vertical – the method did not want to generate dipping interfaces (Auken & Christiansen, 2004). The inclusion of diagonal spatial differences presented here attempts to remedy this problem.

Smith et al. (1999) and de Groot-Hedlin & Constable (2004) for MT, and Auken & Christiansen (2004) for DC resistivity, have developed 2D inversion procedures which generate blocky, pseudo-layered models. In their approaches, the depths of the horizontal cell boundaries are parameters in the inversion, as well as the conductivities in the cells. Constraints are placed on the lateral variability of the conductivities and the cell boundary depths. This is different from the approach taken here, which is to extend the classic minimum-structure, under-determined inversion algorithm.

## Theory

**General minimum-structure inversion strategy**

Typical minimum-structure, under-determined inversion: (*i*) the subsurface is discretized using a mesh with many cells, (*ii*) mesh kept fixed during inversion, (*iii*) values of the physical property in the cells are the model parameters to be found via the inversion, (*iv*) seek the model which minimizes a combination of data misfit and model complexity.

The composite function to be minimized:

$$\Phi \;=\; \phi_d + \beta\, \phi_m.$$

Measure of data misfit:

$$\phi_d \;=\; \phi_d(\mathbf{u}), \qquad \mathbf{u} = \mathbf{W}_d(\mathbf{d}^{\text{obs}} - \mathbf{d}^{\text{prd}}),$$

where $\mathbf{d}^{\text{obs}}$ is the vector of observations, $\mathbf{d}^{\text{prd}}$ is the vector of data computed for the model, and $\mathbf{W}_d$ is a diagonal matrix of the reciprocals of the measurement uncertainties.

Measure of structure in the model:

$$\phi_m \;=\; \sum_k \alpha_k\, \phi_k(\mathbf{v}_k), \qquad \mathbf{v}_k = \mathbf{W}_k(\mathbf{m} - \mathbf{m}_k^{\text{ref}}),$$

where $\mathbf{m}$ is the vector of model parameters, $\mathbf{W}_k$ are finite-difference matrices, and $\mathbf{m}_k^{\text{ref}}$ are possible reference models.

## Theory (continued)

**General measures**

The general form of the measures $\phi_d$ and $\phi_k$ is:

$$\phi(\mathbf{x}) \;=\; \sum_{j=1}^{N} \rho(x_j).$$

There are numerous possibilities: (*i*) the classic $l_2$ norm with

$$\rho(x) \;=\; x^2,$$

(*ii*) the $l_p$ norm with

$$\rho(x) \;=\; |x|^p,$$

(*iii*) the $M$-measure of Huber for which

$$\rho(x) \;=\; \begin{cases} x^2 & |x| \le c, \\ 2c|x| - c^2 & |x| > c, \end{cases}$$

and (*iv*) the perturbed $l_p$ norm-like measure of Ekblom for which

$$\rho(x) \;=\; \left(x^2 + \varepsilon^2\right)^{p/2}.$$

Because of the squaring, minimizing the $l_2$ norm of a vector suppresses elements which are much larger than their neighbours. Variation of the sizes of the elements gets smeared out as much as possible.

Minimizing an $l_1$-type measure of a vector does not penalize against isolated large elements.

Hence, minimizing an $l_1$ measure of spatial differences of a model allows for large, localized differences. This produces models with sharp interfaces between relatively uniform zones.

(See appendix below for the iterative solution procedure.)

**Measure of model structure**

In typical 2D minimum-structure inversions, the measure of model structure comprises a term involving finite differences in the $x$-direction and a term involving finite differences in the $z$-direction (and a diagonal weighting matrix). When $l_2$ measures are used, these two terms constitute the norm of the gradient of the model. When $l_1$ measures are used, the terms work independently to generate vertical and horizontal interfaces. To generate dipping interfaces, it is suggested here that terms involving finite differences in the diagonal directions be included (see Figure 1).

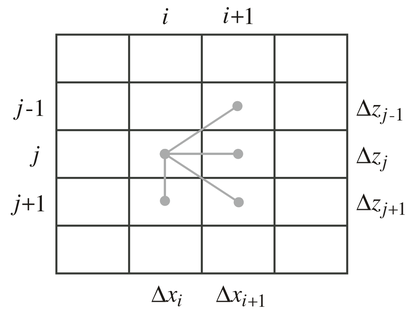


Figure 1: A sub-section of the 2D model mesh showing the possible directions of the finite differences.

The measure of model structure $\phi_m$ therefore comprises five terms, with $\mathbf{W}_1$ a diagonal weighting matrix, $\mathbf{W}_2$ and $\mathbf{W}_3$ the usual $x$ and $z$ finite-difference matrices, and $\mathbf{W}_4$ and $\mathbf{W}_5$ finite-difference operators for diagonally up to the right and diagonally down to the right.

## Example

A synthetic MT data-set was generated from the conductivity model shown in Figure 2 – real and imaginary parts of the E-polarization impedances at 17 locations at three frequencies (3, 10 and 30 Hz). Gaussian random noise of standard deviation equal to 1% of a datum was added.

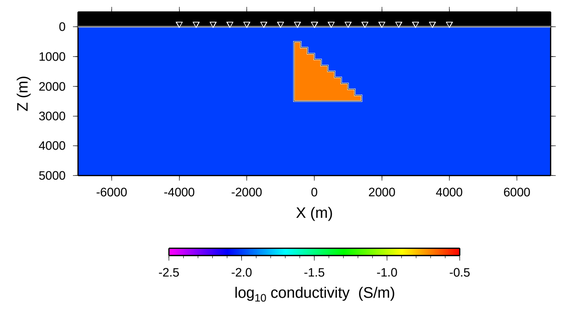


Figure 2: The conductivity model from which the synthetic data-set was generated. The triangles indicate the observation locations.

Three inversion results are shown below. In each inversion, the trade-off parameter was started at a relatively large value (100), and then halved at each iteration until a value was reached which resulted in a misfit sufficiently close to the target value. The trade-off parameter was then kept constant until the model and the objective function were no longer changing.

Figure 3 shows the model produced using the usual $l_2$ measures, and the usual $x$ and $z$ finite-difference terms. Figure 4 shows the model produced using the Ekblom measure with $p$=1, and only $x$ and $z$ difference terms. Figure 5 shows the model produced using the Ekblom measure ($p$=1) and with all four directions of finite differences.

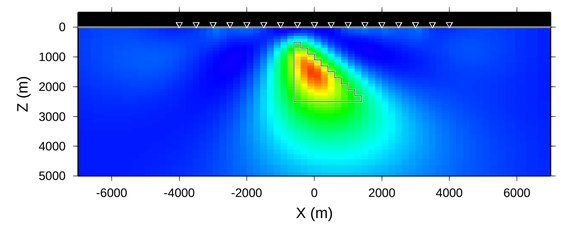


Figure 3: The model produced using $l_2$ measure of model structure.

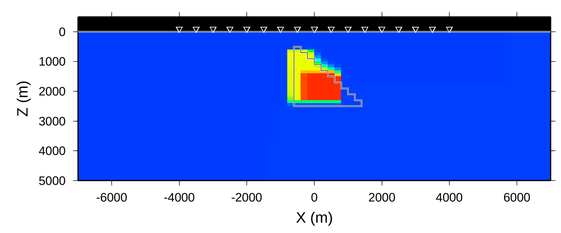


Figure 4: The model generated using an $l_1$-type measure of model structure with the usual $x$ and $z$ finite-difference terms (i.e., no explicit diagonal differences).

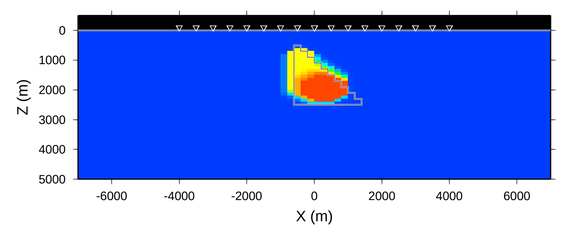


Figure 5: The model generated using an $l_1$-type measure of model structure and diagonal finite differences.

## Appendix: Iterative solution procedure

The inverse problem is non-linear, both because of the non-linear relationship between data and model, and because of the non-$l_2$ measures. So use iterative procedure, with linearized approximation of the relationship between data and model. At the $n$th iteration, minimize

$$\Phi^n \;=\; \phi_d^n + \beta^n\, \phi_m^n,$$

with

$$\phi_d^n \;=\; \phi_d(\mathbf{u}), \qquad \mathbf{u} = \mathbf{W}_d(\mathbf{d}^{\text{obs}} - \mathbf{d}^{n-1} - \mathbf{J}\,\delta\mathbf{m}),$$

$$\mathbf{d}^n \;\approx\; \mathbf{d}^{n-1} + \mathbf{J}\,\delta\mathbf{m}, \qquad J_{ij} \;=\; \frac{\partial d_i}{\partial m_j},$$

and

$$\phi_m^n \;=\; \sum_k \alpha_k \phi_k(\mathbf{v}_k), \qquad \mathbf{v}_k = \mathbf{W}_k(\mathbf{m}^{n-1} + \delta\mathbf{m} - \mathbf{m}_k^{\text{ref}}),$$

where $\mathbf{m}^{n-1}$ is the model from the previous iteration, $\mathbf{d}^{n-1}$ is the data for that model, $\delta\mathbf{m} = \mathbf{m}^n - \mathbf{m}^{n-1}$, and $\mathbf{J}$ is the Jacobian matrix of sensitivities computed for the model from the previous iteration.

Differentiate $\Phi^n$ with respect to $\delta m_k$, and equate resulting derivatives to zero. Differentiating the general form of the measures gives:

$$\frac{\partial \phi(\mathbf{x})}{\partial\, \delta m_k} \;=\; \sum_{j=1}^{N} \rho'(x_j) \frac{\partial x_j}{\partial\, \delta m_k}, \quad \textit{i.e.,} \qquad \frac{\partial \phi(\mathbf{x})}{\partial\, \delta\mathbf{m}} \;=\; \mathbf{B}^T\mathbf{q},$$

where $\partial\phi/\partial\,\delta\mathbf{m} = \left(\partial\phi/\partial\,\delta m_1, \ldots, \partial\phi/\partial\,\delta m_N\right)^T$, $B_{ij} = \partial x_i/\partial\,\delta m_j$, and $\mathbf{q} = \left(\rho'(x_1), \ldots, \rho'(x_N)\right)^T$. Introducing the diagonal matrix

$$\mathbf{R} \;=\; \text{diag}\left\{\rho'(x_1)/x_1, \ldots, \rho'(x_N)/x_N\right\}$$

gives

$$\frac{\partial \phi(\mathbf{x})}{\partial\, \delta\mathbf{m}} \;=\; \mathbf{B}^T\mathbf{R}\mathbf{x}.$$

For the measure of misfit, $\mathbf{B}$ is $-\mathbf{W}_d\mathbf{J}$. For the measure of model structure, $\mathbf{B}$ is $\mathbf{W}_k$. For Ekblom's measure, the elements of $\mathbf{R}$ are:

$$R_{ii} \;=\; p\left(x_i^2 + \varepsilon^2\right)^{p/2-1}.$$

The linear system of equations to be solved at each iteration is therefore:

$$\left[\mathbf{J}^T\mathbf{W}_d^T\mathbf{R}_d\mathbf{W}_d\mathbf{J} + \beta^n \sum_k \alpha_k \mathbf{W}_k^T\mathbf{R}_k\mathbf{W}_k\right] \delta\mathbf{m}$$
$$= \mathbf{J}^T\mathbf{W}_d^T\mathbf{R}_d\mathbf{W}_d\left(\mathbf{d}^{\text{obs}} - \mathbf{d}^{n-1}\right) + \beta^n \sum_k \mathbf{W}_k^T\mathbf{R}_k\mathbf{W}_k\left(\mathbf{m}_i^{\text{ref}} - \mathbf{m}^{n-1}\right).$$

The matrices $\mathbf{R}$ and the Jacobian matrix are updated at each iteration. This is know as iteratively reweighted least squares (IRLS).

For 2D MT example presented here, forward modelling done using finite difference procedure (Farquharson et al., 2002), operation of Jacobian matrix, or its transpose, on a vector done using the forward solver (e.g. Mackie and Madden, 1993, Rodi and Mackie, 2001), linear system of equations solved using direct LU decomposition or conjugate gradients.

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